Quantum limits of sub-Laplacians via joint spectral calculus

Cyril Letrouit

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Cyril LETROUIT*†

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Abstract

We establish two results concerning the Quantum Limits (QLs) of some sub-Laplacians. First, under a commutativity assumption on the vector fields involved in the definition of the sub-Laplacian, we prove that it is possible to split any QL into several pieces which can be studied separately, and which come from well-characterized parts of the associated sequence of eigenfunctions.

Secondly, building upon this result, we classify all QLs of a particular family of sub-Laplacians defined on products of compact quotients of Heisenberg groups. We express the QLs through a disintegration of measure result which follows from a natural spectral decomposition of the sub-Laplacian in which harmonic oscillators appear.

Both results are based on the construction of an adequate elliptic operator commuting with the sub-Laplacian, and on the associated joint spectral calculus. They illustrate the fact that, because of the possibly high degeneracy of the spectrum, the spectral theory of sub-Laplacians can be very rich.

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1 Introduction and main results

1.1 Motivation

The main goal of this paper is to establish some properties of the eigenfunctions of families of hypoelliptic operators in the high-frequency limit. A typical problem is the description of the Quantum Limits (QL) of the operator, i.e., the measures which are weak limits of a

*Sorbonne Université, Université Paris-Diderot SPC, CNRS, Inria, Laboratoire Jacques-Louis Lions, équipe CAGE, F-75005 Paris (`letrouit@ljll.math.upmc.fr`)

†DMA, École normale supérieure, CNRS, PSL Research University, 75005 Paris

subsequence of squares of eigenfunctions. All the operators we consider in the sequel are sub-Laplacians, and they are in particular hypoelliptic.

We briefly recall the general definition of a sub-Laplacian. Let $n \in \mathbb{N}^*$ and let M be a smooth connected compact manifold of dimension n without boundary. We consider a smooth vector distribution $\mathcal{D}$ on M (possibly with non-constant rank), and a Riemannian metric g on $\mathcal{D}$. We denote by $\mathcal{D}_x$ the distribution at point $x \in M$. We assume that $\mathcal{D}$ satisfies the Hörmander condition

$$\mathrm{Lie}(\mathcal{D}) = TM \tag{1}$$

(see [Mon02]). Let μ be a smooth volume form on M and let $\Delta_{g,\mu}$ be the selfadjoint sub-Laplacian associated with the metric g and with the volume form μ. If $\mathcal{D}$ is locally spanned by N smooth vector fields $X_1, \ldots, X_N$ that are g-orthonormal, then we set

$$\Delta_{g,\mu} = -\sum_{i=1}^{N} X_i^* X_i = \sum_{i=1}^{N} (X_i^2 + \mathrm{div}_\mu(X_i) X_i) \tag{2}$$

where the star designates the transpose in $L^2(M, \mu)$. This definition does not depend on the choice of the g-orthonormal frame $X_1, \ldots, X_N$. We can also note that if $\mathcal{D} = TM$, g is a Riemannian metric on TM and μ is the canonical volume on (M, g), then $\Delta_{g,\mu}$ is the usual Laplace-Beltrami operator.

Under the assumption (1), $\Delta_{g,\mu}$ is hypoelliptic (see [Hör67]), has a compact resolvent, and there exists a sequence of (real-valued) eigenfunctions of $-\Delta_{g,\mu}$ associated to the eigenvalues in increasing order $0 = \lambda_1 < \lambda_2 \leq \ldots$ (with $\lambda_j \to +\infty$ as $j \to +\infty$) which is orthonormal for the $L^2(M, \mu)$ scalar product. The main purpose of this paper is to understand the possible behaviours of the sequence of probability measures $|\varphi_k|^2 d\mu$ when $(\varphi_k)_{k \in \mathbb{N}^*}$ is a sequence of normalized eigenfunctions of $-\Delta_{g,\mu}$ with associated eigenvalue tending to $+\infty$, for particular sub-Laplacians $\Delta_{g,\mu}$, typically by describing its weak limits (in the sense of duality with continuous functions).

There is a phase-space extension of these weak limits whose behaviour is also of interest. Let us recall the following definition (see [Gér91b]):

Definition 1. *Let $(u_k)_{k \in \mathbb{N}^*}$ be a bounded sequence in $L^2(M)$ and weakly converging to 0. We call microlocal defect measure of $(u_k)_{k \in \mathbb{N}^*}$ any Radon measure ν on S^*M such that for any $a \in \mathscr{S}^0(M)$, there holds*

$$(Op(a) u_{\sigma(k)}, u_{\sigma(k)}) \underset{k \to +\infty}{\longrightarrow} \int_{S^*M} a \, d\nu$$

for some extraction σ. Here, $(\cdot, \cdot)$ denotes the $L^2(M, \mu)$ scalar product, $\mathscr{S}^0(M)$ is the space of classical symbols of order 0, and $Op(a)$ is the Weyl quantization of a (see Appendix A).

Microlocal defect measures are useful tools for studying the (asymptotic) concentration and oscillation properties of sequences, and they are necessarily non-negative.

Definition 2. *Given a sequence $(\varphi_k)_{k \in \mathbb{N}^*}$ of eigenfunctions of $-\Delta_{g,\mu}$ with $\|\varphi_k\|_{L^2(M,\mu)} = 1$, we call Quantum Limit (QL) any microlocal defect measure of $(\varphi_k)_{k \in \mathbb{N}^*}$.*

Remark 3. *Since φ_k, $k \in \mathbb{N}^*$ is normalized, any QL is a probability measure on S^*M.*

For any Riemannian manifold (M, g), it is well known that any Quantum Limit ν of the Laplace-Beltrami operator Δ_g is invariant under the geodesic flow $\exp(t\vec{H})$: there holds $\exp(t\vec{H})\nu = 0$ for any $t \in \mathbb{R}$. To see it, we note that for any sequence $(\varphi_k)_{k \in \mathbb{N}^*}$ consisting of normalized eigenfunctions of $-\Delta_g$, there holds

$$(\exp(-it\sqrt{-\Delta_g}) \mathrm{Op}(a) \exp(it\sqrt{-\Delta_g}) \varphi_k, \varphi_k)_{L^2} = (\mathrm{Op}(a)\varphi_k, \varphi_k)_{L^2} \tag{3}$$

for any $t \in \mathbb{R}$, any $k \in \mathbb{N}^*$ and any classical symbol $a \in \mathscr{S}^0(M)$. It follows from Egorov's theorem that $\exp(-it\sqrt{-\Delta_g}) \mathrm{Op}(a) \exp(it\sqrt{-\Delta_g})$ is a pseudodifferential operator of order 0 with principal symbol $a \circ \exp(t\vec{H})$, which in turn implies $\exp(t\vec{H})\nu = 0$.

The structure and the invariance properties of the Quantum Limits of sub-Laplacians is more complicated than that of Riemannian Laplacians. To see it, let us consider a general sub-Laplacian $\Delta_{g,\mu}$, the principal symbol $g^* = \sigma_P(-\Delta_{g,\mu})$, and the associated sub-Riemannian geodesic flow $\vec{g}^*$. The invariance of Quantum Limits of $\Delta_{g,\mu}$ under the sub-Riemannian geodesic flow $\vec{g}^*$ is still true, but it does not say anything about the part of the QL lying in $(g^*)^{-1}(0)$ since the geodesic flow is stationary at such points. Indeed, we note that the above computation (3) does not work anymore for general sub-Laplacians since $\sqrt{-\Delta_{g,\mu}}$ is not a pseudodifferential operator near its characteristic manifold $(g^*)^{-1}(0)$, and hence Egorov's theorem does not apply.

Therefore, it is interesting to determine other invariance properties for this part of the QL. In [CdVHT18, Theorem B], it was proved that for any sub-Laplacian $\Delta_{g,\mu}$, any of its Quantum Limit ν can be decomposed as a sum $\nu = \nu_0 + \nu_\infty$ of mutually singular measures, where ν_0 is supported in the "elliptic part" of $(g^*)^{-1}(\mathbb{R} \setminus 0)$ and is invariant under the sub-Riemannian geodesic flow $\vec{g}^*$, and ν_∞ is supported in $(g^*)^{-1}(0)$ (and its invariance properties are far more difficult to establish, as will be seen below). It was also proved that for "most" QLs, $\nu_0 = 0$, and therefore most our efforts in this paper are devoted to understand ν_∞. The precise statement of [CdVHT18, Theorem B] is recalled in Proposition 11 below.

Remark 4. *The point of view taken in this paper is definitely Euclidean, meaning that we do not use pseudodifferential calculus adapted to the stratified Lie algebra which possibly shows up while studying sub-Laplacians. However, our results share connexions with important problems in non-commutative Fourier analysis, which are explained in Section 1.7.*

1.2 A commutativity assumption

The description of the Quantum Limits of general sub-Laplacians is a very difficult problem, since even for Riemannian Laplacians it is far from being understood (see Section 1.7).

In this paper, we restrict our attention to particular sub-Laplacians, for which, despite their lack of ellipticity, techniques of joint (elliptic) spectral calculus apply thanks to additional commutativity assumptions.

Let us fix a sub-Riemannian structure $(M, \mathcal{D}, g)$ satisfying (1), a smooth g-orthonormal frame $X_1, \ldots, X_N$, a smooth volume μ on M, and consider the associated sub-Laplacian $\Delta_{g,\mu}$ given by (2). We make the following assumption:

Assumption (A). There exist $Z_1, \ldots, Z_m$ smooth global vector fields on M such that:

(i) At any point $x \in M$ where $\mathcal{D}_x \neq T_xM$, the vector fields $Z_1(x), \ldots, Z_m(x)$ complete $\mathcal{D}_x$ into a basis of T_xM (in particular, they are independent);

(ii) For any $1 \leq i, j \leq m$, there holds $[\Delta_{g,\mu}, Z_i] = [Z_i, Z_j] = 0$.

Assumption (A) is satisfied for example in the following cases:

- For H-type sub-Laplacians (whose definition is recalled in Appendix B.1, see also see [Kap80] and [FKF20]), in particular for sub-Laplacians defined on quotients of the $(2d+1)$-dimensional Heisenberg group (see Appendix B.1). In this case, the vector fields Z_j form a basis of the center of the associated Lie algebra.
- For the quasi-contact sub-Laplacian $\partial_x^2 + (\partial_y - x\partial_z)^2 + \partial_w^2$ defined on $\mathbf{H} \times (\mathbb{R}/2\pi\mathbb{Z})$, where $\mathbf{H}$ is a quotient of the 3D Heisenberg group (see Section 1.5 for a precise definition). In this case, $m = 1$ and $Z_1 = \partial_z$.
- For manifolds obtained as products of the previous examples (and associated sub-Laplacians obtained by sum), since Assumption (A) is stable by product.
- More generally, let us consider a connected, simply connected nilpotent Lie group G which is stratified of step 2, in the sense that its left-invariant Lie algebra $\mathfrak{g}$, assumed to be real-valued and of finite dimension, is endowed with a vector space decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ where $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z} \neq \{0\}$ and $\mathfrak{z}$ is the center of $\mathfrak{g}$. The exponential map $\exp : G \to \mathfrak{g}$, which is a diffeomorphism, allows to identify G and $\mathfrak{g}$. Fixing a scalar product

$\langle\cdot,\cdot\rangle$ on $\mathfrak{v}$, there exists an orthonormal basis of left-invariant vector fields $X_1,\ldots,X_N$ spanning $\mathfrak{v}$. The associated sub-Laplacian is $\Delta_{\mathfrak{g}}=\sum_{i=1}^N X_i^2$, which can also be defined on any compact left-quotient H of G. Then, taking for the Z_j a basis of $\mathfrak{z}$, we see that Assumption (A) is satisfied. This setting encompasses all previous examples.

- For Baouendi-Grushin-type sub-Laplacians: e.g., for $\partial_x^2+\sin(x)^2\partial_y^2$ in $(\mathbb{R}/2\pi\mathbb{Z})^2$, the set of points x such that $\mathcal{D}_x\neq T_xM$ consists of the singular lines $\{x=0\}$ and $\{x=\pi\}$, and we can take $Z_1=\partial_y$. Note that the usual Baouendi-Grushin operator is $\partial_x^2+x^2\partial_y^2$, but we put here a sine in order to define it on a compact manifold without boundary.
- All the above examples are "step 2", but it is also possible to build ad hoc sub-Laplacians satisfying Assumption (A) and requiring higher-order brackets of the X_i to generate the whole tangent bundle, see Appendix B.2.

1.3 The cotangent bundle T^*M under Assumption (A)

Let us introduce a few notations. We set $g^*=\sigma_P(-\Delta_{g,\mu})$ where σ_P denotes the principal symbol of a pseudodifferential operator (see Appendix A), and we denote by $\Sigma=(g^*)^{-1}(0)=\mathcal{D}^\perp\subset T^*M$ the characteristic cone (where $\perp$ is in the sense of duality). This is the region of the phase-space where $\Delta_{g,\mu}$ is not elliptic: in some sense, it is the region which is of most interest in the study of sub-Laplacians, in contrast with usual Riemannian Laplacians. We make the identification

$$S^*M=U^*M\cup S\Sigma \tag{4}$$

where S^*M is the cosphere bundle (i.e., the sphere bundle of T^*M), $U^*M=\{g^*=1\}$ is a cylinder bundle and $S\Sigma$ is a sphere bundle consisting of the points at infinity of the compactification of U^*M.

We denote by ω the canonical symplectic form on the cotangent bundle T^*M of M. In local coordinates (q,p) of T^*M, we have $\omega=dq\wedge dp$. Given a smooth Hamiltonian function $h:T^*M\to\mathbb{R}$, we denote by $\vec{h}$ the corresponding Hamiltonian vector field on T^*M, defined by $\iota_{\vec{h}}\omega=dh$. Given any smooth vector field V on M, we denote by h_V the Hamiltonian function (momentum map) on T^*M associated with V, defined in local coordinates by $h_V(q,p)=p(V(q))$. The Hamiltonian flow $\exp(t\vec{h}_V)$ of h_V projects onto the integral curves of V.

In all the sequel, we consider a sub-Laplacian $\Delta_{g,\mu}$ satisfying Assumption (A). Let $\mathcal{P}$ be the set of all subsets of $\{1,\ldots,m\}$. We write Σ as a disjoint union

$$\Sigma=\bigcup_{\mathcal{J}\in\mathcal{P}}\Sigma_{\mathcal{J}} \tag{5}$$

where, for $\mathcal{J}\in\mathcal{P}$, $\Sigma_{\mathcal{J}}$ is defined as the set of points $(q,p)\in\Sigma$ with

$$\big(h_{Z_j}(q,p)\neq 0\big)\Leftrightarrow(j\in\mathcal{J}).$$

Note that (5) is a disjoint union and that the $\Sigma_{\mathcal{J}}$ are non-empty, thanks to point (i) in Assumption (A) together with the fact that $\pi(\Sigma)=\{x\in M,\ \mathcal{D}_x\neq T_xM\}$ where $\pi:T^*M\to M$ is the canonical projection.

1.4 Quantum Limits under Assumption (A)

Our first main result states that it is possible to split any QL into several pieces which can be studied separately, and which come from well-characterized parts of the associated sequence of eigenfunctions. In order to give a precise statement, we need to define joint microlocal defect measures:

Definition 5. *Let $(u_k)_{k\in\mathbb{N}^*},(v_k)_{k\in\mathbb{N}^*}$ be bounded sequences in $L^2(M)$ such that u_k and v_k weakly converge to 0 as $k\to+\infty$. We call joint microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$ and $(v_k)_{k\in\mathbb{N}^*}$ any Radon measure ν_{joint} on S^*M such that for any $a\in\mathscr{S}^0(M)$, there holds*

$$(Op(a)u_{\sigma(k)},v_{\sigma(k)})\underset{k\to+\infty}{\longrightarrow}\int_{S^*M}a\,d\nu_{joint}$$

for some extraction σ.

In case $u_k = v_k$ for any $k \in \mathbb{N}^*$, we recover the microlocal defect measures of Definition 1. Note that joint microlocal defect measures are not necessarily non-negative, and that joint Quantum Limits (defined as joint microlocal defect measures of two sequences of eigenfunctions) are not necessarily invariant under the geodesic flow, even in the Riemannian case.

Our first main result is the following.

Theorem 1. *Let $\Delta_{g,\mu}$ satisfy Assumption (A). We assume that $(\varphi_k)_{k\in\mathbb{N}^*}$ is a normalized sequence of eigenfunctions of $-\Delta_{g,\mu}$ with associated eigenvalues $\lambda_k \to +\infty$. Then, up to extraction of a subsequence, one can decompose*

$$\varphi_k = \varphi_k^{\emptyset} + \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \varphi_k^{\mathcal{J}}, \tag{6}$$

with the following properties:

- *The sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ has a unique Quantum Limit ν;*
- *For any $\mathcal{J} \in \mathcal{P}$ and any $k \in \mathbb{N}^*$, $\varphi_k^{\mathcal{J}}$ is an eigenfunction of $-\Delta_{g,\mu}$ with eigenvalue λ_k;*
- *Using the identification $S^*M = U^*M \cup S\Sigma$ (see (4)), the sequence $(\varphi_k^{\emptyset})_{k\in\mathbb{N}^*}$ admits a unique microlocal defect measure $\beta\nu^{\emptyset}$, where $\beta \in [0,1]$, $\nu^{\emptyset} \in \mathscr{P}(S^*M)$ and $\nu^{\emptyset}(S\Sigma) = 0$, and, for any $\mathcal{J} \in \mathcal{P}\setminus\{\emptyset\}$, the sequence $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ also admits a unique microlocal defect measure $\nu^{\mathcal{J}}$, having all its mass contained in $S\Sigma_{\mathcal{J}}$;*
- *For any $\mathcal{J} \neq \mathcal{J}' \in \mathcal{P}$, the joint microlocal defect measure of the sequences $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ and $(\varphi_k^{\mathcal{J}'})_{k\in\mathbb{N}^*}$ vanishes. As a consequence,*

$$\nu = \beta\nu^{\emptyset} + \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \nu^{\mathcal{J}} \tag{7}$$

  and the sum in (7) is supported in $S\Sigma$.

In this statement, we separated the emptyset from the other subsets $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$ to emphasize on the concentration of $\beta\nu^{\emptyset}$ on U^*M, while the rest of the measure ν in (7) is supported in $S\Sigma$. This is purely artificial, since one could have included $\beta\nu^{\emptyset}$ into the sum over $\mathcal{J}$. Besides, the notation $\nu^{\emptyset}$ used above corresponds to the notation ν_0 in [CdVHT18] (see Proposition 11 below): we changed it to get a unified notation for the different parts of the QL, namely $\nu^{\emptyset}$ and $\nu^{\mathcal{J}}$.

Theorem 1 follows from joint spectral calculus (see [RS72, VII and VIII.5]) for the operators $Z_1, \ldots, Z_m$ and $-\Delta_{g,\mu}$ which is made possible thanks to Assumption (A). The ideas underlying Theorem 1 are close to those of [CdV79, Theorem 0.6], which deals with the joint spectrum of commuting pseudodifferential operators whose sum of squares is elliptic, see Remark 16. Theorem 1 might also be adapted to more general settings, and we plan to study further applications in a future work.

1.5 Products of flat contact sub-Laplacians

Our second main result gives much more information on Quantum Limits, but it works only for a very specific family of sub-Laplacians, which in particular satisfy Assumption (A). In order to define these sub-Laplacians, let us first recall the definition of the 3D Heisenberg group. Endow $\mathbb{R}^3$ with the product law

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' - xy').$$

With this law, $\widetilde{\mathbf{H}} = (\mathbb{R}^3, \star)$ is a Lie group, which is isomorphic to the group of matrices

$$\left\{ \begin{pmatrix} 1 & x & -z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},\ x, y, z \in \mathbb{R} \right\}$$

endowed with the standard product law on matrices.

We consider the left quotient $\mathbf{H} = \Gamma\backslash\widetilde{\mathbf{H}}$ where $\Gamma = (\sqrt{2\pi}\mathbb{Z})^2 \times 2\pi\mathbb{Z}$ is a cocompact subgroup of $\widetilde{\mathbf{H}}$ (meaning that $\mathbf{H}$ is compact). Note that $\mathbf{H}$ is not homeomorphic to $\mathbb{T}^2 \times \mathbb{S}^1$ since its fundamental group is Γ. The vector fields on $\mathbf{H}$

$$X = \partial_x \quad \text{and} \quad Y = \partial_y - x\partial_z$$

are left invariant, and we consider $\Delta_{\mathbf{H}} = X^2 + Y^2$ the associated sub-Laplacian (here μ is the Lebesgue measure $\mu = dxdydz$ and (X, Y) is orthonormal for g).

Then, we consider the product manifold $\mathbf{H}^m$ and the associated sub-Laplacian Δ for some integer $m \geq 2$, that is

$$\Delta = \Delta_{\mathbf{H}} \otimes (\mathrm{Id})^{\otimes m-1} + \mathrm{Id} \otimes \Delta_{\mathbf{H}} \otimes (\mathrm{Id})^{m-2} + \ldots + (\mathrm{Id})^{\otimes m-1} \otimes \Delta_{\mathbf{H}}, \tag{8}$$

which is a second-order pseudodifferential operator. Below, we give an expression (9) for Δ which is more tractable. In the sequel, we fix once for all an integer $m \geq 2$.

Remark 6. *If $(\varphi_k)_{k\in\mathbb{N}^*}$ denotes an orthonormal Hilbert basis of $L^2(\mathbf{H})$ consisting of eigenfunctions of $-\Delta_{\mathbf{H}}$, then*

$$\{\varphi_{k_1} \otimes \ldots \otimes \varphi_{k_m} \mid k_1, \ldots, k_m \in \mathbb{N}^*\}$$

is an orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$. However, there exist orthonormal Hilbert bases of $L^2(\mathbf{H}^m)$ which cannot be put in this tensorized form.

In this introductory section, the sub-Laplacian we consider is either $\Delta_{\mathbf{H}}$, or Δ, or an arbitrary sub-Laplacian $\Delta_{g,\mu}$ on a general sub-Riemannian manifold $(M, \mathcal{D}, g)$. In all cases, we keep the same notations g^*, Σ and $S\Sigma$ to denote the objects introduced in Section 1.3, without any reference in the notation to the underlying manifold even for the particular sub-Laplacians $\Delta_{\mathbf{H}}$ and Δ. It should not lead to any confusion since the context is precisely stated when necessary.

In order to give a precise statement of our second main result, it is necessary to introduce a decomposition of the sub-Laplacian Δ defined by (8). Taking coordinates (x_j, y_j, z_j) on the j-th copy of $\mathbf{H}$, we can write

$$\Delta = \sum_{j=1}^{m} (X_j^2 + Y_j^2) \tag{9}$$

with $X_j = \partial_{x_j}$ and $Y_j = \partial_{y_j} - x_j\partial_{z_j}$. We note that Δ satisfies Assumption (A) (for $Z_j = \partial_{z_j}$ for $j = 1, \ldots, m$).

Let us briefly describe Σ for the sub-Laplacian Δ. Denoting by (q, p) the canonical coordinates in $T^*\mathbf{H}^m$, i.e., $q = (x_1, y_1, z_1, \ldots, x_m, y_m, z_m)$ and $p = (p_{x_1}, p_{y_1}, p_{z_1}, \ldots, p_{x_m}, p_{y_m}, p_{z_m})$, we obtain that

$$\Sigma = \left\{(q, p) \in T^*\mathbf{H}^m \mid p_{x_j} = p_{y_j} - x_j p_{z_j} = 0 \text{ for any } 1 \leq j \leq m\right\},$$

which is isomorphic to $\mathbf{H}^m \times \mathbb{R}^m$. Above any point $q \in \mathbf{H}^m$, the fiber of Σ is of dimension m, and therefore, above any point $q \in \mathbf{H}^m$, $S\Sigma$ consists of an $(m-1)$-dimensional sphere.

For $1 \leq j \leq m$, we consider the operator $R_j = \sqrt{\partial_{z_j}^* \partial_{z_j}}$ and we make a Fourier expansion with respect to the z_j-variable in the j-th copy of $\mathbf{H}$. On the eigenspaces corresponding to non-zero modes of this Fourier decomposition, we define the operator $\Omega_j = -R_j^{-1}\Delta_j = -\Delta_j R_j^{-1}$ where $\Delta_j = X_j^2 + Y_j^2$. For example, $-\Delta$ acts as

$$-\Delta = \sum_{j=1}^{m} R_j \Omega_j \tag{10}$$

on any eigenspace of $-\Delta$ on which $R_j \neq 0$ for any $1 \leq j \leq m$. Moreover, R_j and Ω_j are pseudodifferential operators of order 1 in any cone of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j} = 0\}$ is reduced to 0.

The operator Ω_j, seen as an operator on the j-th copy of $\mathbf{H}$, is an harmonic oscillator, having in particular eigenvalues $2n+1$, $n \in \mathbb{N}$ (see [CdVHT18, Section 3.1]). Moreover, the operators Ω_i (considered this time as operators on $\mathbf{H}^m$) commute with each other and with the operators R_j.

Writing Σ as a disjoint union (5), we notice that $\Sigma_{\mathcal{J}}$ is indeed the set of points $(q,p) \in \Sigma$ with $p = (p_{x_1}, p_{y_1}, p_{z_1}, \ldots, p_{x_m}, p_{y_m}, p_{z_m})$ such that

$$\left(p_{z_j} \neq 0\right) \Leftrightarrow (j \in \mathcal{J}).$$

For $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, we consider the simplex

$$\mathbf{S}_{\mathcal{J}} = \left\{ s = (s_j) \in \mathbb{R}_+^{\mathcal{J}},\ \sum_{j \in \mathcal{J}} s_j = 1 \right\}$$

and, for $s = (s_j) \in \mathbf{S}_{\mathcal{J}}$ and $(q,p) \in \Sigma_{\mathcal{J}}$, we set

$$\rho_s^{\mathcal{J}}(q,p) = \sum_{j \in \mathcal{J}} s_j |p_{z_j}|.$$

Note that we have

$$\rho_s^{\mathcal{J}}(q,p) = (\sigma_P(R_s))_{|\Sigma_{\mathcal{J}}} \quad \text{where} \quad R_s = \sum_{j \in \mathcal{J}} s_j R_j \tag{11}$$

where σ_P denotes the principal symbol (see Appendix A). Moreover, the Hamiltonian vector field $\vec{\rho}_s^{\mathcal{J}}$ is well-defined on $\Sigma_{\mathcal{J}}$ and smooth.

Remark 7. *Projecting the flow of $\vec{\rho}_s^{\mathcal{J}}$ on M, we obtain straight lines described by changes proportional to s_j in the z_j coordinates, for $j \in \mathcal{J}$. Once all coordinates x_i, y_i (for $1 \leq i \leq m$) and z_i (for $i \notin \mathcal{J}$) have been fixed - since they are preserved by the flow -, these straight lines are similar to the lines given by the geodesic flow on the flat $|\mathcal{J}|$-dimensional Riemannian torus in the variables z_j (for $j \in \mathcal{J}$).*

Finally, denoting by $\mathscr{M}_+(E)$ (respectively $\mathscr{P}(E)$) the set of non-negative Radon measures (respectively Radon probability measures) on a given separated space E, we set[1]

$$\begin{aligned}\mathscr{P}_{S\Sigma} = \Bigg\{ \nu_\infty = \sum_{\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}} \nu^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m), \quad \nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu_s^{\mathcal{J}} dQ^{\mathcal{J}}(s), \\ \text{where } Q^{\mathcal{J}} \in \mathscr{M}_+(\mathbf{S}_{\mathcal{J}}),\ \nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m), \\ \nu_s^{\mathcal{J}}(S^*\mathbf{H}^m \setminus S\Sigma_{\mathcal{J}}) = 0 \text{ and, for } Q^{\mathcal{J}}\text{-almost any } s \in \mathbf{S}_{\mathcal{J}}\ ,\ \vec{\rho}_s^{\mathcal{J}} \nu_s^{\mathcal{J}} = 0 \Bigg\}.\end{aligned} \tag{12}$$

This last definition means that for any continuous function $a : S\Sigma \to \mathbb{R}$, there holds

$$\int_{S\Sigma} a d\nu_\infty = \sum_{\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}} \int_{\mathbf{S}_{\mathcal{J}}} \left(\int_{S\Sigma_{\mathcal{J}}} a d\nu_s^{\mathcal{J}} \right) dQ^{\mathcal{J}}(s).$$

[1] The notation $S\Sigma_{\mathcal{J}}$ which appears for example in (12) designates in all the sequel the set of points (q,p) of $S\Sigma$ which have null (homogeneous) coordinate p_{z_i} for any $i \notin \mathcal{J}$ and non-null p_{z_j} for $j \in \mathcal{J}$. Note that this set is, in general, neither open nor closed.

In a few words, (12) means that any measure $\nu_\infty \in \mathscr{P}_{S\Sigma}$ is supported in $S\Sigma$, and that its invariance properties are given separately on each set $S\Sigma_{\mathcal{J}}$ (for $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$). Its restriction to any of these sets, denoted by $\nu^{\mathcal{J}}$, can be disintegrated with respect to $\mathbf{S}_{\mathcal{J}}$, and for any $s \in \mathbf{S}_{\mathcal{J}}$, there is a corresponding measure $\nu_s^{\mathcal{J}}$ which is invariant by the flow $e^{t\vec{\rho}_s^{\mathcal{J}}}$.

Our second main result is the following:

Theorem 2. *Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be an orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ labeled in increasing order. Let ν be a Quantum Limit associated to the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$. Then, using the identification (4), we can write ν as the sum of two mutually singular measures $\nu = \beta\nu^\emptyset + (1-\beta)\nu_\infty$, with $\nu^\emptyset, \nu_\infty \in \mathscr{P}(S^*\mathbf{H}^m)$, $\beta \in [0,1]$ and*

(1) $\nu^\emptyset(S\Sigma) = 0$ and $\nu^\emptyset$ is invariant under the sub-Riemannian geodesic flow $e^{t\vec{g}^*}$;

(2) $\nu_\infty \in \mathscr{P}_{S\Sigma}$.

Moreover, there exists a density-one sequence $(k_\ell)_{\ell\in\mathbb{N}}$ of positive integers such that, if ν is a QL associated with a subsequence of $(k_\ell)_{\ell\in\mathbb{N}}$, then the support of ν is contained in $S\Sigma$, i.e., $\beta = 0$ in the previous decomposition.

The reason why we consider here only orthonormal bases is to give a sense to the density-one subsequence of the last part of the statement. However, the first part of the statement is true for any sequence of normalized eigenfunctions of $-\Delta$ with eigenvalues tending to $+\infty$.

Note that Theorem 2 holds for *any* orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$, and not only for the bases described in Remark 6.

Also, the sub-Riemannian geodesic flow $e^{t\vec{g}^*}$ involved in Theorem 2 is completely integrable, see [ABB19, Chapter 18].

The converse of Theorem 2 holds too, in the following sense:

Theorem 3. *Let $\nu_\infty \in \mathscr{P}_{S\Sigma}$. Then ν_∞ is a Quantum Limit associated to some sequence of normalized eigenfunctions of $-\Delta$ with eigenvalues tending to $+\infty$.*

Together, Theorem 2 and Theorem 3 yield a classification of (nearly) all Quantum Limits of Δ.

Remark 8. *The exact converse of Theorem 2 would guarantee that all measures $\nu \in \mathscr{P}(S^*\mathbf{H}^m)$ of the form $\nu = \beta\nu^\emptyset + (1-\beta)\nu_\infty$ with the same assumptions on β, $\nu^\emptyset$ and ν_∞ as in Theorem 2 are Quantum Limits. Our statement is weaker since it does not say anything about the measures ν for which $\beta \neq 0$ (which are rare, as stated in Theorem 2), but we do not think that a stronger converse statement for Theorem 2 holds.*

Remark 9. *Theorems 2 and 3 remain true for slightly more general sub-Laplacians than those considered here. Indeed, for any $d \in \mathbb{N}$, one can consider the (2d+1)-dimensional Heisenberg group $\widetilde{\mathbf{H}}_d$ and its quotient $\mathbf{H}_d = \Gamma_d \backslash \widetilde{\mathbf{H}}_d$ by the discrete cocompact subgroup $\Gamma_d = (\sqrt{2\pi}\mathbb{Z})^{2d} \times 2\pi\mathbb{Z}$. Then, one can define as in Section 1.5 a natural sub-Laplacian $\Delta_{\mathbf{H}_d}$ on $\mathbf{H}_d$ (see Appendix C). Given a finite sequence of positive integers $d_1, \ldots, d_m$, one can consider the associated sub-Laplacian on $\mathbf{H}_{d_1} \times \ldots \times \mathbf{H}_{d_m}$ defined as in (8). Then, Theorems 2 and 3 are still true in this setting (mutatis mutandis). However, for the sake of clarity of presentation, we found it preferable to write full details only in the case $d_1 = \ldots = d_m = 1$, since it already contains the key ideas.*

Remark 10. *The problem of identifying other families of sub-Laplacians for which a full characterization of QLs is possible is open; it requires to identify a family of 1-homogeneous Hamiltonians on Σ replacing the family $(\rho_s^{\mathcal{J}})$. E.g., for the quasi-contact sub-Laplacian $\partial_x^2 + (\partial_y - x\partial_z)^2 + \partial_w^2$, defined on $\mathbf{H} \times (\mathbb{R}/2\pi\mathbb{Z})$, it does not seem possible to identify such a family because of the additional ∂_w^2 term which is separated from the $R\Omega$-factorization of the rest of the sub-Laplacian.*

1.6 Comments on the main results

In order to explain the contents of Theorem 2 and Theorem 3, we recall the following result, which is valid for any sub-Laplacian $\Delta_{g,\mu}$.

Proposition 11. *[CdVHT18, Theorem B] Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be an orthonormal Hilbert basis of $L^2(M,\mu)$ consisting of eigenfunctions of $-\Delta_{g,\mu}$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ labeled in increasing order. Let ν be a QL associated with $(\varphi_k)_{k\in\mathbb{N}^*}$. Using the identification $S^*M = U^*M \cup S\Sigma$ (see (4)), the probability measure ν can be written as the sum $\nu = \beta\nu_0 + (1-\beta)\nu_\infty$ of two mutually singular measures with $\nu_0, \nu_\infty \in \mathscr{P}(S^*M)$, $\beta \in [0,1]$ and*

(1) $\nu_0(S\Sigma) = 0$ and ν_0 is invariant under the sub-Riemannian geodesic flow $\vec{g}^*$;

*(2) ν_∞ is supported on $S\Sigma$. Moreover, **in the 3D contact case**, ν_∞ is invariant under the lift to $S\Sigma$ of the Reeb flow.*[2]

Moreover, there exists a density-one sequence $(k_\ell)_{\ell\in\mathbb{N}}$ of positive integers such that, if ν is a QL associated with a subsequence of $(k_\ell)_{\ell\in\mathbb{N}}$, then the support of ν is contained in $S\Sigma$, i.e., $\beta = 0$ in the previous decomposition.[3]

The last part of Proposition 11 shows that ν_∞ is the "main part" of the QL, but, according to Point (2), its invariance properties were known only in the 3D contact case. Theorem 3 and Point (2) of Theorem 2 serve as substitutes to Point (2) of Proposition 11 for the sub-Laplacians Δ on $\mathbf{H}^m$.

Compared to the invariance properties of the QLs of 3D contact sub-Laplacians described in Proposition 11, the invariance property described by Point (2) of Theorem 2 involves an infinite number of different Hamiltonian vector fields $\vec{\rho}_s^{\mathcal{J}}$ on $S\Sigma$.

Spectrum of $-\Delta$. The particularly rich structure of the Quantum Limits of the sub-Laplacian $-\Delta$ described in Theorem 2 is due to the high degeneracy of its spectrum. To make an analogy with the Riemannian case, the QLs of the usual flat Riemannian torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ have a rich structure (see [Jak97]), whereas the QLs of irrational Riemannian tori are much simpler to describe.

Recall that the spectrum $\mathrm{sp}(-\Delta_{\mathbf{H}})$ is given by

$$\begin{aligned}\mathrm{sp}(-\Delta_{\mathbf{H}}) = \{\lambda_{\ell,\alpha} = (2\ell+1)|\alpha| \ \mid\ \ell\in\mathbb{N},\ \alpha\in\mathbb{Z}\setminus\{0\}\}\\ \cup\,\{\mu_{k_1,k_2} = 2\pi(k_1^2+k_2^2) \ \mid\ (k_1,k_2)\in\mathbb{Z}^2\}\end{aligned}$$

where $\lambda_{\ell,\alpha}$ is of multiplicity $|\alpha|$, multiplied by the number of decompositions of $\lambda_{\ell,\alpha}$ into the form $(2\ell'+1)|\alpha'|$ (see [CdVHT18, Proposition 3.1]). Therefore, using a tensorial orthonormal Hilbert basis of $L^2(\mathbf{H}^m)$ consisting of eigenfunctions of $-\Delta$, we get that

$$\mathrm{sp}(-\Delta) = \left\{\sum_{j=1}^{J}(2n_j+1)\,|\alpha_j| + 2\pi\sum_{i=1}^{2(m-J)} k_i^2 \text{ with } 0\le J\le m,\ k_i\in\mathbb{Z},\ n_j\in\mathbb{N},\ \alpha_j\in(\mathbb{Z}\setminus\{0\})\right\}$$

(see Section 3.2 for a detailed proof) and the multiplicities in $\mathrm{sp}(-\Delta)$ can be deduced from those in $\mathrm{sp}(-\Delta_{\mathbf{H}})$. Note that the eigenvalues for which $J = m$ form a density-one subsequence of all eigenvalues labeled in increasing order.

The specific algebraic structure of $\mathrm{sp}(-\Delta)$ will be exploited in particular to prove Theorem 3.

[2]See [CdVHT18] for a definition of the Reeb flow, or Appendix C.

[3]The proof of this last fact follows from the results in [CdVHT18], although it is not explicitely stated there. Let us sketch the proof. By [CdVHT18, Proposition 4.3], we know that the microlocal Weyl measure of $\Delta_{g,\mu}$ is supported in $S\Sigma$. It then follows from [CdVHT18, Corollary 4.1] that for every $A\in\Psi^0(M)$ whose principal symbol vanishes on Σ, there holds $V(A)=0$, where $V(A)$ is the variance introduced in [CdVHT18, Definition 4.1]. Finally, following the proof of Theorem B(2) in [CdVHT18], we get the result.

Remark 12. *Contrarily to those of flat tori (see [Jak97]), the Quantum Limits of* $\mathbf{H}^m$ *(or, more precisely, their pushforward under the canonical projection onto* $\mathbf{H}^m$*) are not necessarily absolutely continuous. It was already remarked in the case* $m = 1$ *in [CdVHT18, Proposition 3.2(2)].*

Remark 13. *There is no clear link of our result with the concept of "second microlocalization", although such a link may seem possible at first sight. Focusing on a Quantum Limit supported in* $S\Sigma$*, our study builds upon a* spectral *decomposition of it, and not upon a second direction of microlocalization as is usually done while studying fine properties of sequences of solutions of an operator (see for example [FK00]).*

1.7 Related problems and bibliographical comments.

Quantum Limits of Riemannian Laplacians. The study of Quantum Limits for Riemannian Laplacians is a long-standing question. Over the years, a particular attention has been drawn towards Riemannian manifolds whose geodesic flow is ergodic since in this case, up to extraction of a density-one subsequence, the set of Quantum Limits is reduced to the Liouville measure, a phenomenon which is called Quantum Ergodicity (see for example [Shn74], [CdV85], [Zel87]). For compact arithmetic surfaces, a detailed study of invariant measures lead to the resolution of the Quantum Unique Ergodicity conjecture for these manifolds, meaning that the extraction of a density-one subsequence in the previous result is even not necessary for these particular manifolds ([Lin06]). In manifolds which have a degenerate spectrum, the set of Quantum Limits is generally richer: see for example [Jak97] for the description of Quantum Limits on flat tori or [ALM16] for the case of the disk. Also, the Quantum Limits of the sphere $\mathbb{S}^d$ equipped with its canonical metric (see [JZ96]) have been fully characterized. However, to the author's knowledge, few papers until now have been devoted to the study of Quantum Limits of product of Riemannian manifolds (see [HPT20, Corollary 2] for a recent result).

Quantum Limits of sub-Laplacians. The understanding of Quantum Limits of general sub-Laplacians remains a largely unexplored question. Their study was undertaken in the work [CdVHT18], which was mainly devoted to the 3D contact case - encompassing for example the case of the manifold $\mathbf{H}$ - although some results are valid for any sub-Laplacian (see Proposition 11 of the present paper). The authors proved Weyl laws (i.e., results "in average" on eigenfunctions), a result of decomposition of Quantum Limits, and also Quantum Ergodicity properties (i.e., equidistribution of Quantum Limits under an ergodicity assumption) for 3D contact sub-Laplacians. The Quantum Limits of H-type (or Heisenberg-type) sub-Laplacians were also implicitly studied in [FKF20] thanks to a detailed study of the Schrödinger flow: the authors developed a notion of semiclassical measures adapted to "Heisenberg type" sub-Laplacians thanks to non-commutative Fourier analysis and a subsequent adapted definition of pseudodifferential operators. Taking in Theorem 2.10(ii)(2) of [FKF20] eigenfunctions of the sub-Laplacian as initial data of the Schrödinger equation yields a decomposition of Quantum Limits which may be regarded as an analog of Theorem 2 in the context of H-type groups (more precisely, one should use the adaptation to the compact (quotient) setting of these results which was done in [FKL20], among other things); however, the result of [FKF20] is proved by totally different techniques, and in particular the splitting of Quantum Limits which we obtain through joint spectral calculus (see below) is replaced in [FKF20] by non-commutative harmonic analysis.

Non-commutative harmonic analysis. As already mentioned in Remark 4, it is possible to use the stratified Lie algebra structure to study the spectral theory of (nilpotent) sub-Laplacians, as done for example in [FKF20]. This work builds upon non-commutative harmonic analysis (see [Tay86]) to develop a pseudodifferential calculus and semiclassical tools "naturally attached to the sub-Laplacian". It is very likely that one could have given a proof of Theorems 2 and 3 based on similar tools as in [FKF20]. The point of view we adopt in the present paper is different: it only requires "classical" pseudodifferential calculus (briefly recalled in Appendix A) since there is still enough commutativity and ellipticity from the choice of operators under study. Beside making the results more accessible to some readers, it allows us to isolate in each eigenfunction the piece which is responsible, in the high-frequency limit, for a given part of the QL. Moreover, our method only builds upon abstract commutation arguments, at least for Theorem 1, and in particular it avoids the computation of irreducible representations which are always specific to certain families of groups (e.g., H-type groups in [FKF20]).

Part of our results can be reinterpreted through the light of noncommutative harmonic analysis. For example, the part of the QL in U^*M, namely $\beta\nu^{\emptyset}$ (see (7)), is described in [FKF20] as the part of the semiclassical measure supported above the finite dimensional representations $\pi_x^{0,\omega}$ (see [FKF20, Section 2.2.1]), and the fact that $\beta\nu^{\emptyset} = 0$ for "almost all" QLs (see Proposition 11) can be recovered from the fact that the Plancherel measure denoted by $|\lambda|^d d\lambda$ in [FKF20] gives no mass to finite-dimensional representations.

Also, in the setting covered by Theorems 2 and 3, i.e., products of quotients of the Heisenberg group, the joint spectrum of $(\Delta_1, \ldots, \Delta_m, i^{-1}\partial_{z_1}, \ldots, i^{-1}\partial_{z_m})$, which can be drawn in $\mathbb{R}^{2m}$, is called "Heisenberg fan". This terminology was introduced in [Str91] for the 3D Heisenberg sub-Laplacian; in our case, this fan consists in a discrete set of points which can be gathered into lines (see [Str91, Figure 1]). In case $m = 1$, the subset of points (or joint eigenvalues) corresponding to $\varphi_k^{\emptyset}$ and $\nu^{\emptyset}$ in the statement of Theorem 2 can be seen as points close to the vertical line $\{0\} \times \mathbb{R} \subset \mathbb{R}^2$. Similar descriptions can be given in case $m \geq 2$. Also, let us mention that we could derive from the proof of Theorem 1 a generalization of the definition of the Heisenberg fan to any sub-Laplacian satisfying Assumption (A), as the joint spectrum of $(-\Delta_{g,\mu}, |Z_1|, \ldots, |Z_m|)$.

Let us also mention that sub-Laplacians on products of Heisenberg groups (and, more generally, on "decomposable groups") were analysed in [BFKG] with a non-commutative harmonic analysis point of view in order to establish Strichartz estimates (see notably [BFKG, Section 1.4 and Corollary 1.6]).

The particular geometry of the QLs of Δ. As already recalled, the QLs of Riemannian Laplacians are invariant by the geodesic flow: in some sense, this means that for any $(x, \xi) \in T^*M$, the QL near (x, ξ) "is invariant in the direction given by ξ". The above Proposition 11 for 3D contact sub-Laplacians, and the result of [FKF20, Theorem 2.10(ii)(2)] for H-type groups mentioned in this section extend this intuition to these sub-Laplacians. But Theorems 2 and 3 show that such a property is not true for any sub-Laplacian: there exist QLs of Δ and points $(x, \xi) \in \mathbf{H}^m$ such that the QL near (x, ξ) is not invariant in the direction ξ, but in some other direction of the cotangent bundle (parametrized by $s \in \mathbf{S}$). This fact will be highlighted again along the proof of Theorem 3.

Joint spectral calculus. A key ingredient in the proof of all results of the present paper is the joint spectral calculus (see [RS72, VII and VIII.5] and [CdV79]) associated to the operators $Z_1, \ldots, Z_m$ and $-\Delta_{g,\mu}$. This joint calculus, at least for Heisenberg groups, is well-known, see for example [DS84, Section 2], or [Tha09] for the quotient case. It was used for instance in [MRS95] to prove a Marcinkiewicz multiplier theorem in H-type groups.

Structure of the paper. In Section 2 we prove Theorem 1 using joint spectral calculus. Then, Section 3 is devoted to the proof of Theorem 2. In Section 3.2, we explain the spectral decomposition of $L^2(\mathbf{H}^m)$ according to the eigenspaces of the harmonic oscillators Ω_j. Building upon this spectral decomposition and Theorem 1, we establish in Section 3.3 Theorem 2. In Section 4, we prove Theorem 3 by constructing explicitely a sequence of eigenfunctions with prescribed Quantum Limit.

In Appendix A, we recall some basic facts of pseudodifferential calculus and two related elementary lemmas. In Appendix B, we provide some supplementary material on Assumption (A). Finally, in Appendix C, we prove a result concerning Quantum Limits of flat contact manifolds in any dimension: for such manifolds, the invariance properties of Quantum Limits are essentially the same as in the 3D case. Although this is a direct consequence of the results in [FKF20], we decided to provide here a short and self-contained proof since this can be seen as a toy model for the averaging techniques used repeatedly in the proof of Theorem 2.

We also mention that in a previous version of this paper[4], we explain an alternative way to obtain the measure $Q^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$ and the family of measures $(\nu_s^{\mathcal{J}})_{s\in\mathbf{S}_{\mathcal{J}}}$ on $S\Sigma_{\mathcal{J}}$, based on pure functional analysis.

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2 Proof of Theorem 1

In this Section, we prove Theorem 1. We fix a sub-Laplacian $\Delta_{g,\mu}$ satisfying Assumption (A), we fix $(\varphi_k)_{k\in\mathbb{N}^*}$ a sequence of eigenfunctions of $-\Delta_{g,\mu}$ associated with the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ with $\lambda_k \to +\infty$ and $\|\varphi_k\|_{L^2} = 1$, and we consider ν, a Quantum Limit associated to the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$.

Let us first give an intuition of how the proof goes. We decompose φ_k as a sum of functions which are joint eigenfunctions of $-\Delta_{g,\mu}$ and of all the $Z_j^*Z_j$ for $1 \leq j \leq m$. Each of these functions is an eigenfunction of $-\Delta_{g,\mu}$ with same eigenvalue λ_k as φ_k. Then, roughly speaking, we gather some of these functions into $\varphi_k^{\emptyset}$ or into $\varphi_k^{\mathcal{J}}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, depending on their eigenvalues with respect to the operators $Z_j^*Z_j$ (for $1 \leq j \leq m$) and $-\Delta_{g,\mu}$. More precisely, setting

$$E = -\Delta_{g,\mu} + \sum_{j=1}^{m} Z_j^* Z_j \in \Psi^2(M), \tag{13}$$

the functions which we select (asymptotically as $k \to +\infty$) to be in $\varphi_k^{\mathcal{J}}$ are those such that:

1. $-\Delta_{g,\mu} \ll E$;
2. if $i \notin \mathcal{J}$, then $Z_j^* Z_j \ll E$;
3. if $j \in \mathcal{J}$, then $Z_j^* Z_j \gtrsim E$.

Here, since we consider joint eigenfunctions of $-\Delta_{g,\mu}$, E and $Z_j^*Z_j$ for any $1 \leq j \leq m$, the above notation $A \ll B$ (resp. $A \gtrsim B$) means that the eigenvalue with respect to A is negligible compared to (resp. is greater than a constant times) the eigenvalue with respect to B.

This splitting "quantizes" the fact that $\Sigma_{\mathcal{J}}$ is the set of points (q,p) of T^*M for which $g^*(q,p) = 0$ (point 1 above) and $h_{Z_j(q,p)}$ is non-nul if and only if $j \in \mathcal{J}$ (points 2 and 3 above).

Here is the rigorous proof:

Proof of Theorem 1. For $n \in \mathbb{N}^*$, let $\chi_n \in C_c^\infty(\mathbb{R}, [0,1])$ such that $\chi_n(x) = 1$ for $|x| \leq \frac{1}{2n}$ and $\chi_n(x) = 0$ for $|x| \geq \frac{1}{n}$. We consider E given by (13) which, thanks to point (i) in Assumption (A), is elliptic. Its principal symbol is

$$\sigma_P(E) = g^* + \sum_{j=1}^{m} \sigma_P(Z_j^* Z_j).$$

[4] `https://arxiv.org/pdf/2007.00910.pdf`

Also, thanks to point (ii) in Assumption (A), we know that E commutes with Z_j, for any $1 \leq j \leq m$, and with $-\Delta_{g,\mu}$. Therefore, thanks to functional calculus (see [RS72, VII and VIII.5]), for $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, it makes sense to consider the operator

$$P_n^{\mathcal{J}} = \chi_n\left(\frac{-\Delta_{g,\mu}}{E}\right) \prod_{i \notin \mathcal{J}} \chi_n\left(\frac{Z_i^* Z_i}{E}\right) \prod_{j \in \mathcal{J}} (1-\chi_n)\left(\frac{Z_j^* Z_j}{E}\right). \tag{14}$$

Similarly, we consider

$$P_n^{\emptyset} = (1-\chi_n)\left(\frac{-\Delta_{g,\mu}}{E}\right). \tag{15}$$

As we will see, for any $\mathcal{J} \in \mathcal{P}$, $P_n^{\mathcal{J}} \in \Psi^0(M)$ and, as $n \to +\infty$, its principal symbol tends either to the characteristic function $\mathbf{1}_{\Sigma_{\mathcal{J}}} : T^*M \to \mathbb{R}$ of $\Sigma_{\mathcal{J}}$, or to the characteristic function $\mathbf{1}_{U^*M}$ of U^*M if $\mathcal{J} = \emptyset$. Recall that $\Sigma_{\mathcal{J}}$ has been defined in (5).

For any $\mathcal{J} \in \mathcal{P}$, the following properties hold:

(1) $P_n^{\mathcal{J}} \in \Psi^0(M)$;
(2) $[P_n^{\mathcal{J}}, \Delta_{g,\mu}] = 0$;
(3) If $\mathcal{J} \neq \emptyset$, then $\sigma_P(P_n^{\mathcal{J}}) \to \mathbf{1}_{\Sigma_{\mathcal{J}}}$ pointwise as $n \to +\infty$.
If $\mathcal{J} = \emptyset$, then $\sigma_P(P_n^{\mathcal{J}}) \to \mathbf{1}_{U^*M}$ pointwise as $n \to +\infty$.

Let us prove Point (1). Since $E \in \Psi^2(M)$ is elliptic, it is invertible, and thus $-\Delta_{g,\mu}E^{-1} = -E^{-1}\Delta_{g,\mu} \in \Psi^0(M)$ is self-adjoint. Hence, by [HV00, Theorem 1(ii)], $(1-\chi_n)\left(\frac{-\Delta_{g,\mu}}{E}\right) \in \Psi^0(M)$ with principal symbol

$$(1-\chi_n)\left(\frac{g^*}{\sigma_P(E)}\right).$$

Similarly, the operators $\chi_n\left(\frac{-\Delta_{g,\mu}}{E}\right)$, $\chi_n\left(\frac{Z_i^* Z_i}{E}\right)$ and $(1-\chi_n)\left(\frac{Z_j^* Z_j}{E}\right)$ (for any $1 \leq i, j \leq m$) belong to $\Psi^0(M)$ with respective principal symbols

$$\chi_n\left(\frac{g^*}{\sigma_P(E)}\right), \qquad \chi_n\left(\frac{|h_{Z_i}|^2}{\sigma_P(E)}\right) \quad \text{and} \quad (1-\chi_n)\left(\frac{|h_{Z_j}|^2}{\sigma_P(E)}\right).$$

Hence, $P_n^{\mathcal{J}} \in \Psi^0(\mathbf{H}^m)$.

Point (2) is an immediate consequence of functional calculus, since $\Delta_{g,\mu}$ commutes with E and with Z_j for any $1 \leq j \leq m$.

Let us prove Point (3). For $\kappa > 0$, we consider the cone

$$S_\kappa := \left\{ \frac{g^*}{\sigma_P(E)} \leq \kappa \right\} \subset T^*M$$

and, for $1 \leq j \leq m$, we also consider the cone

$$T_\kappa^j = \left\{ \frac{|h_{Z_j}|^2}{\sigma_P(E)} \leq \kappa \right\} \subset T^*M.$$

For the moment, we assume $\mathcal{J} \neq \emptyset$. Then, the support of $\sigma_P(P_n^{\mathcal{J}})$ is contained in $S_{\frac{1}{n}}$, in $T_{\frac{1}{n}}^i$ for $i \notin \mathcal{J}$ and in the complementary set $(T_{\frac{1}{2n}}^j)^c$ for $j \in \mathcal{J}$. It follows that, in the limit $n \to +\infty$, $\sigma_P(P_n^{\mathcal{J}})$ vanishes everywhere outside the set of points (q,p) satisfying $g^*(q,p) = 0$,

$$h_{Z_i}(q,p) = 0, \quad \forall i \notin \mathcal{J}$$
$$h_{Z_j}(q,p) \neq 0, \quad \forall j \in \mathcal{J}.$$

We note that these relations exactly define the set $\Sigma_{\mathcal{J}}$.
Conversely, let $(q,p) \in \Sigma_{\mathcal{J}}$. Our goal is to show that $\sigma_P(P_n^{\mathcal{J}})(q,p) = 1$ for sufficiently large $n \in \mathbb{N}^*$. It follows from a separate analysis of the principal symbol of each factor in the product (14):

- Since $(q,p) \in \Sigma$, there holds $g^*(q,p) = 0$, hence

$$\chi_n\left(\frac{g^*}{\sigma_P(E)}\right) = 1;$$

- For $i \notin \mathcal{J}$, since $h_{Z_i}(q,p) = 0$, there holds

$$\chi_n\left(\frac{|h_{Z_i}|^2}{\sigma_P(E)}\right)(q,p) = 1;$$

- For $j \in \mathcal{J}$, we know that $h_{Z_j}(q,p) \neq 0$. Hence, for n sufficiently large, at (q,p),

$$(1-\chi_n)\left(\frac{|h_{Z_j}|^2}{\sigma_P(E)}\right)(q,p) = 1.$$

All in all, $\sigma_P(P_n^{\mathcal{J}})(q,p) = 1$ for sufficiently large n, which proves Point (3) in case $\mathcal{J} \neq \emptyset$. The proof in the case $\mathcal{J} = \emptyset$ is very similar, for the sake of brevity we do not repeat it here.

We now conclude the proof of Theorem 1. We consider, for fixed $n \in \mathbb{N}$ and $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, the sequence $(P_n^{\mathcal{J}}\varphi_k)_{k\in\mathbb{N}^*}$, which, thanks to Points (1) and (2), is also a sequence of eigenfunctions of $-\Delta_{g,\mu}$ with same eigenvalues as φ_k. We denote by $\nu_n^{\mathcal{J}}$ a microlocal defect measure of $(P_n^{\mathcal{J}}\varphi_k)_{k\in\mathbb{N}^*}$ and by $\nu_n^{\emptyset}$ a microlocal defect measure of the sequence given by the eigenfunctions

$$\varphi_k - \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} P_n^{\mathcal{J}}\varphi_k.$$

Furthermore, we can assume thanks to the diagonal extraction process that the extraction used to obtain all these microlocal defect measures is the same for any $n \in \mathbb{N}^*$ and any $\mathcal{J} \in \mathcal{P}$.

Finally, we take $\nu^{\mathcal{J}}$ a weak-star limit of $(\nu_n^{\mathcal{J}})_{n\in\mathbb{N}}$ and $\beta\nu^{\emptyset}$ a weak-star limit of $(\nu_n^{\emptyset})_{n\in\mathbb{N}}$, with $\nu \in \mathscr{P}(S^*M)$ and $\beta \in [0,1]$. Thanks to the analysis done while proving Point (3), we know that $\nu^{\mathcal{J}}$ gives no mass to the complementary of $S\Sigma_{\mathcal{J}}$ in S^*M, and that $\nu^{\emptyset}(S\Sigma) = 0$. Again, thanks to the diagonal extraction process, up to extraction of a subsequence in $k \in \mathbb{N}^*$, we can write

$$\varphi_k = \varphi_k^{\emptyset} + \sum_{\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}} \varphi_k^{\mathcal{J}} \tag{16}$$

where the unique microlocal defect measure of $(\varphi_k^{\emptyset})_{k\in\mathbb{N}^*}$ is $\beta\nu^{\emptyset}$, and $\varphi_k^{\mathcal{J}} = P_{r(k)}^{\mathcal{J}}\varphi_k$ (for some function r tending (slowly) to $+\infty$ as $k \to +\infty$) has a unique microlocal defect measure as $k \to +\infty$, which is $\nu^{\mathcal{J}}$.

Let us prove that (16) implies (7). For that, we first recall an elementary lemma concerning joint microlocal defect measures (see Definition 5). It is proved in Appendix A.

Lemma 14. *Let $(u_k), (v_k)$ be two sequences of functions weakly converging to 0, each with a unique microlocal defect measure, which we denote respectively by μ_{11} and μ_{22}. Then, any joint microlocal defect measures μ_{12} (resp. μ_{21}) of $(u_k)_{k\in\mathbb{N}^*}$ and $(v_k)_{k\in\mathbb{N}^*}$ (resp. of $(v_k)_{k\in\mathbb{N}^*}$ and $(u_k)_{k\in\mathbb{N}^*}$) is absolutely continuous with respect to both μ_{11} and μ_{22}.*

Using Lemma 14, we then notice that if $\mathcal{J}, \mathcal{J}' \in \mathcal{P} \setminus \{\emptyset\}$ are distinct, the joint microlocal defect measures of $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ and $(\varphi_k^{\mathcal{J}'})_{k\in\mathbb{N}^*}$ vanish since $\Sigma_{\mathcal{J}}$ and $\Sigma_{\mathcal{J}'}$ are disjoint. Similarly, the joint microlocal defect measure of $(\varphi_k^0)_{k\in\mathbb{N}^*}$ with the sequence $(\varphi_k^{\mathcal{J}})_{k\in\mathbb{N}^*}$ vanishes for any $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. Therefore, evaluating $(\mathrm{Op}(a)\varphi_k, \varphi_k)$ and using (16), we obtain (7), which finishes the proof of Theorem 1. $\square$

Remark 15. *The above proof is inspired by the proof of a slightly different fact (see [Gér91a, Proposition 3.3]): if θ is the unique microlocal defect measure of a sequence $(\psi_k)_{k\in\mathbb{N}^*}$ of functions over a manifold M, A (resp. B) is a closed (resp. open) subset of S^*M, and A and B form a partition of S^*M, then we can write $\theta = \theta_A + \theta_B$, with θ_A (resp. θ_B)*

supported in A (resp. $\theta_B(A)=0$) and $\psi_k=\psi_k^A+\psi_k^B$ such that θ_A (resp. θ_B) is a microlocal defect measure of $(\psi_k^A)_{k\in\mathbb{N}^*}$ (resp. of $(\psi_k^B)_{k\in\mathbb{N}^*}$). The proof just consists in choosing symbols $p_n\in\mathscr{S}^0(M)$ concentrating on A and taking $\psi_k^A=Op(p_n)\psi_k$ as in the proof above.

In the proof of Theorem 1, we had to choose particular symbols p_n in order to ensure that $\varphi_k^{\mathcal{J}}$ and $\varphi_k^{\emptyset}$ are still eigenfunctions of $-\Delta_{g,\mu}$.

Remark 16. *As already mentioned, the ideas underlying Theorem 1 are close to those of [CdV79, Theorem 0.6], which deals with the joint spectrum of commuting pseudodifferential operators whose sum of squares is elliptic. The parallel is the following: the elliptic operator Q in [CdV79] is replaced here by E, and the operators P_i in [CdV79] are replaced here by the X_i and the Z_j.*

With this parallel in mind and using the tools developed in the above proof, given a Riemannian Laplacian $\Delta_g=\sum X_i^2$ with all the X_i commuting and a sequence of eigenfunctions of Δ_g, one could identify which part of the eigenfunctions concentrates on each part of the cotangent bundle.

In our setting, not all X_i and Z_j commute, but $\sum_{i=1}^N X_i^*X_i$ commutes with all Z_j, which is sufficient because we do not look for any information on the QLs in U^*M. Our statement is, in some sense, more precise than [CdV79, Theorem 0.6] since the splitting of eigenfunctions is made precise, but also less general because X_i and Z_j are differential, and not general pseudodifferential operators as in [CdV79].

3 Proof of Theorem 2

This section is devoted to the proof of Theorem 2. Therefore, we fix $m\geq 2$ and $\Delta_{g,\mu}=\Delta$ as in Section 1.5. The last part of Theorem 2 is an immediate consequence of the last part of Proposition 11, and therefore we are reduced to prove Points (1) and (2). The first step in the proof consists in reducing the analysis to the part of the QL above $\Sigma_{\mathcal{J}}$ for some $\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}$, and it is achieved thanks to Theorem 1 as follows.

Reduction to a fixed $\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}$. Combining Theorem 1 with Point (1) of Proposition 11, we see that it is enough to prove Point (2) of Theorem 2, and that it is possible to assume that $(\varphi_k)_{k\in\mathbb{N}^*}$ is a sequence of eigenfunctions with eigenvalue tending to $+\infty$, and with a unique microlocal defect measure ν, which can be assumed to be supported in $S\Sigma$. Indeed, thanks to Theorem 1, we can even assume that all the mass of ν is contained in $S\Sigma_{\mathcal{J}}$ for some $\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}$, i.e., $\nu=\nu^{\mathcal{J}}$: once we have established the decomposition

$$\nu^{\mathcal{J}}=\int_{\mathbf{S}_{\mathcal{J}}}\nu_s^{\mathcal{J}}\,dQ^{\mathcal{J}}(s),$$

Point (2) of Theorem 2 follows by just gluing all pieces of ν together thanks to Theorem 1.

Therefore, in order to establish Point (2) of Theorem 2, we assume that the unique microlocal defect measure of $(\varphi_k)_{k\in\mathbb{N}^*}$ has no mass outside $S\Sigma_{\mathcal{J}}$ for some $\mathcal{J}\in\mathcal{P}\setminus\{\emptyset\}$. By symmetry, we can even assume that $\mathcal{J}=\{1,\ldots,J\}$ with $J=\mathrm{Card}(\mathcal{J})$.

To sum up, the sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ that we consider is no more a general sequence of normalized eigenfunctions with eigenvalues tending to $+\infty$, but it satisfies the following property:

Property 1. *$(\varphi_k)_{k\in\mathbb{N}^*}$ is a bounded sequence of eigenfunctions of $-\Delta$ labeled with increasing eigenvalues tending to $+\infty$, and with unique microlocal defect measure ν. Moreover, there exist $J\leq m$ and $r(k)\to+\infty$ as $k\to+\infty$ such that*

$$\varphi_k=P_{r(k)}^{\mathcal{J}}\varphi_k \tag{17}$$

for $\mathcal{J}=\{1,\ldots,J\}$ and for any $k\in\mathbb{N}^*$, where $P_n^{\mathcal{J}}$ is defined in (14). In particular, ν has no mass outside $S\Sigma_{\mathcal{J}}$.

3.1 Illustration and sketch of proof

Since the rest of the proof is a bit involved, in this section we provide an illustration and a sketch of proof which could be helpful. The proof is written in full details in Sections 3.2 and 3.3. Logically, one may omit the discussion which follows and proceed directly to the next section.

An illustration of Point (2) of Theorem 2. A way to get an intuition of Point (2) of Theorem 2 is to fix $(n_1, \ldots, n_m) \in \mathbb{N}^m$, and to consider a sequence of normalized eigenfunctions $(\psi_k)_{k\in\mathbb{N}^*}$ of $-\Delta$ given in a tensor form as in Remark 6, such that, for any $k \in \mathbb{N}^*$, ψ_k is also, for any $1 \leq j \leq m$, a sequence of eigenfunctions of R_j with eigenvalue tending to $+\infty$, and of Ω_j with eigenvalue $2n_j + 1$. We notice that any associated Quantum Limit ν is supported in $S\Sigma$: it follows directly from the arguments developed in the proof of Theorem 1, since for any $1 \leq j \leq m$, the eigenvalues with respect to R_j^2 are much larger than the eigenvalues with respect to $-\Delta$.

Let $\mathcal{J} = \{1, \ldots, m\} \in \mathcal{P}$. Then, ν is necessarily invariant under the Hamiltonian vector field $\vec{\rho}_s^{\mathcal{J}}$, where $s = (s_1, \ldots, s_m) \in \mathbf{S}_{\mathcal{J}}$ is defined by $s_j = \frac{2n_j+1}{2n_1+1+\ldots+2n_m+1}$ for $j = 1, \ldots, m$. To see it, we set

$$R = \frac{\sum_{j=1}^m (2n_j + 1)R_j}{\sum_{j=1}^m 2n_j + 1}$$

and we note that for any $A \in \Psi^0(\mathbf{H}^m)$, we have

$$([A, R]\psi_k, \psi_k) = (AR\psi_k, \psi_k) - (A\psi_k, R\psi_k) = 0$$

since ψ_k is an eigenfunction of R. In the limit $k \to +\infty$, taking the principal symbol, we obtain $\int_{S\Sigma}\{a, \rho_s^{\mathcal{J}}\}d\nu = 0$ where $a = \sigma_P(A)$. Since it is true for any $a \in \mathscr{S}^0(\mathbf{H}^m)$, this implies $\vec{\rho}_s^{\mathcal{J}}\nu = 0$. Hence, for such sequences $(\psi_k)_{k\in\mathbb{N}^*}$, any QL verifies $\nu = \nu_s^{\mathcal{J}}$ (which is invariant under $\vec{\rho}_s^{\mathcal{J}}$), $Q^{\mathcal{J}}$ is a Dirac mass on s and $Q^{\mathcal{J}'} = 0$ for $\mathcal{P} \ni \mathcal{J}' \neq \mathcal{J}$.

In some sense, any QL supported on $S\Sigma$ is a linear combination of sequences as in the above example, for different $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$ and different $s \in \mathbf{S}_{\mathcal{J}}$.

Roles of R_j and Ω_j. The operators R_j and Ω_j play a key role in the proofs of Theorem 2 and Theorem 3. As illustrated in the previous paragraph, the operators Ω_j are linked with the parameters $s \in \mathbf{S}_{\mathcal{J}}$: in some sense, once the eigenfunctions have been orthogonally decomposed with respect to the operators R_j and Ω_j (as explained in Section 3.2), the ratios between the Ω_j-s determines the invariance property of the associated Quantum Limits through the parameter s and the Hamiltonian vector field $\vec{\rho}_s^{\mathcal{J}}$. On the other side, the operators R_j 'determine' the microlocal support of the associated Quantum Limits, for example the element $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$ (such that the QL concentrates on $S\Sigma_{\mathcal{J}}$). The next paragraph, which is devoted to a sketch of proof of Theorem 2, will make these intuitions more precise.

Sketch of proof. In order to simplify the presentation, in this sketch of proof, we assume that $\mathcal{J} = \{1, \ldots, m\}$ and we omit this notation (writing for example $\mathbf{S}$ instead of $\mathbf{S}_{\mathcal{J}}$), but the ideas are similar for any $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$.

Le us use the decomposition (10) to write each φ_k as a sum of eigenfunctions of operators of the form $\sum_{j=1}^m (2n_j + 1) R_j$ for some integers $n_1, \ldots, n_m$:

$$\varphi_k = \sum_{(n_1,\ldots,n_m)\in\mathbb{N}^m} \varphi_{k,n_1,\ldots,n_m}, \tag{18}$$

$$\text{with } \Omega_j\varphi_{k,n_1,\ldots,n_m} = (2n_j + 1)\varphi_{k,n_1,\ldots,n_m}, \quad \forall\, 1 \leq j \leq m.$$

We will see in Section 3.2 that the decomposition (18) is orthogonal, and therefore each eigenfunction $\varphi_{k,n_1,\ldots,n_m}$ has the same eigenvalue λ_k as φ_k. Then, we do a careful analysis of this decomposition into modes, which, in the limit $k \to +\infty$, gives the disintegration

$\nu = \int_{\mathbf{S}} \nu_s dQ(s)$. This analysis builds upon a partition of the lattice $\mathbb{N}^m$ into positive cones, each of them gathering together the modes $\varphi_{k,n_1,\ldots,n_m}$ for which the m-tuples

$$\left(\frac{2n_1+1}{2n_1+1+\ldots+2n_m+1}, \ldots, \frac{2n_m+1}{2n_1+1+\ldots+2n_m+1}\right)$$

are approximately the same: each of these positive cones accounts for a small region of the simplex $\mathbf{S}$. If $\mathbb{N}^m$ is partitioned into 2^N positive cones C_ℓ^N (with $0 \leq \ell \leq 2^N - 1$), this gathering defines eigenfunctions

$$\varphi_{k,\ell}^N = \sum_{(n_1,\ldots,n_m)\in C_\ell^N} \varphi_{k,n_1,\ldots,n_m}$$

of $-\Delta$ such that

$$\varphi_k = \sum_{\ell=0}^{2^N-1} \varphi_{k,\ell}^N \tag{19}$$

for any $N \in \mathbb{N}^*$.

Taking a microlocal defect measure ν_ℓ^N in each sequence $(\varphi_{k,\ell}^N)_{k\in\mathbb{N}^*}$ and making $N \to +\infty$ (i.e., taking the limit where the positive cones degenerate to half-lines parametrized by $s \in \mathbf{S}$), we obtain from (19) the disintegration $\nu = \int_{\mathbf{S}} \nu_s dQ(s)$.

Given a certain $s = (s_1, \ldots, s_m) \in \mathbf{S}$, $dQ(s)$ accounts for the relative importance, in the limit $N \to +\infty$, of the eigenfunction $\varphi_{k,\ell(N)}^N$ in the sum (19), where $\ell(N)$ is chosen so that the positive cone $C_{\ell(N)}^N$ converges to the half-line with parameter s as $N \to +\infty$.

The invariance property $\vec{\rho}_s \nu_s = 0$ can be seen from the fact that, for any large N and any $0 \leq \ell \leq 2^N - 1$, each eigenfunction $\varphi_{k,n_1,\ldots,n_m}$ with $(n_1, \ldots, n_m) \in C_\ell^N$ is indeed an eigenfunction of the operator

$$\sum_{i=1}^m \left(\frac{2n_i+1}{2n_1+1+\ldots+2n_m+1}\right) R_i$$

which, by definition of $\varphi_{k,\ell}^N$, is approximately equal to $R_s = s_1R_1 + \ldots + s_mR_m$ if $s = (s_1, \ldots, s_m) \in \mathbf{S}$ denotes the parameter of the limiting half-line of the positive cones C_ℓ^N as $N \to +\infty$. Hence, $\varphi_{k,\ell}^N$ is an approximate eigenfunction of R_s, from which it follows by a classical argument that ν_s is invariant under the Hamiltonian vector field $\vec{\rho}_s$ of $\rho_s = (\sigma_P(R_s))_{|\Sigma}$.

3.2 Spectral decomposition of $-\Delta$

In this section, we start the proof of Theorem 2 with a detailed study of the action of $-\Delta$ on $L^2(\mathbf{H}^m)$, writing it under the form of an orthogonal decomposition of eigenspaces.

Let us recall that, for $1 \leq j \leq m$, we set $R_j = \sqrt{\partial_{z_j}^* \partial_{z_j}}$ and we made a Fourier expansion with respect to the z_j-variable. On the eigenspaces corresponding to non-zero modes of this Fourier decomposition, we defined the operator $\Omega_j = -R_j^{-1}\Delta_j = -\Delta_j R_j^{-1}$ where $\Delta_j = X_j^2 + Y_j^2$. For example, $-\Delta$ acts as

$$-\Delta = \sum_{j=1}^m R_j \Omega_j$$

on any eigenspace of $-\Delta$ on which $R_j \neq 0$ for any $1 \leq j \leq m$. Moreover, R_j and Ω_j are pseudodifferential operators of order 1 in any cone of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j} = 0\}$ is reduced to 0 (for example in small conic neighborhoods of $\Sigma_{\mathcal{J}}$ for $\mathcal{J}$ containing j).

The operator Ω_j, seen as an operator on the j-th copy of $\mathbf{H}$, is an harmonic oscillator, having in particular eigenvalues $2n+1$, $n \in \mathbb{N}$ (see [CdVHT18, Section 3.1]). Moreover, the operators Ω_i (considered this time as operators on $\mathbf{H}^m$) commute with each other and with the operators R_j.

Recall that $\mathcal{P}$ stands for the set of all subsets of $\{1,\ldots,m\}$. We fix $\mathcal{J}\in\mathcal{P}$. In the sequel, we think of $\mathcal{J}$ as the set of j for which $R_j\neq 0$. For $j\in\mathcal{J}$ and $n\in\mathbb{N}$, we denote by $E^j_n\subset L^2(\mathbf{H})$ the eigenspace of Ω_j corresponding to the eigenvalue $2n+1$. For $(n_j)\in\mathbb{N}^{\mathcal{J}}$, we set

$$\mathcal{H}^{\mathcal{J}}_{(n_j)}=F^1\otimes\ldots\otimes F^m\subset L^2(\mathbf{H}^m)$$

where $F^j=E^j_{n_j}$ for $j\in\mathcal{J}$ and $F^j=L^2(\mathbf{H})$ otherwise.

We have the orthogonal decomposition

$$L^2(\mathbf{H}^m)=\bigoplus_{\mathcal{J}\in\mathcal{P}}\bigoplus_{(n_j)\in\mathbb{N}^{\mathcal{J}}}\mathcal{H}^{\mathcal{J}}_{(n_j)}. \tag{20}$$

We can also write the associated decomposition of $-\Delta$:

$$-\Delta=\bigoplus_{\mathcal{J}\in\mathcal{P}}\bigoplus_{(n_j)\in\mathbb{N}^{\mathcal{J}}}H^{\mathcal{J}}_{(n_j)}$$

$$\text{with}\quad H^{\mathcal{J}}_{(n_j)}=\sum_{j\in\mathcal{J}}(2n_j+1)\,R_j-\sum_{i\notin\mathcal{J}}(\partial^2_{x_i}+\partial^2_{y_i}).$$

From this, we deduce

$$\begin{aligned}\mathrm{sp}(-\Delta)&=\bigcup_{\mathcal{J}\in\mathcal{P}}\bigcup_{(n_j)\in\mathbb{N}^{\mathcal{J}}}\mathrm{sp}(H^{\mathcal{J}}_{(n_j)})\\&=\Bigg\{\sum_{j\in\mathcal{J}}(2n_j+1)\,|\alpha_j|+2\pi\sum_{i\notin\mathcal{J}}(k_i^2+\ell_i^2),\\&\qquad\qquad\text{with } k_i,\ell_i\in\mathbb{Z},\ \mathcal{J}\in\mathcal{P},\ n_j\in\mathbb{N},\ \alpha_j\in(\mathbb{Z}\setminus\{0\})\Bigg\}\end{aligned}$$

where sp denotes the spectrum.

3.3 Step 2: End of the proof of Point (2) of Theorem 2

In the sequel, the notation $(\cdot,\cdot)$ stands for the $L^2(\mathbf{H}^m)$ scalar product, and the associated norm is denoted by $\|\cdot\|_{L^2}$.

Positive cones. We set $V=\left(-\frac{1}{2},\ldots,-\frac{1}{2}\right)\in\mathbb{R}^J$ and we consider the quadrant

$$V+\mathbb{R}^J_+=\left\{(x_1,\ldots,x_J)\in\mathbb{R}^J\mid x_j\geq-\frac{1}{2}\text{ for any }1\leq j\leq J\right\}.$$

We now define a series of partitions of $V+\mathbb{R}^J_+$ into positive cones with vertex at V, each of these partitions (indexed by N) being composed of 2^N thin positive cones, with the property that each partition is a refinement of the preceding one.

More precisely, these positive cones $C^N_\ell\subset V+\mathbb{R}^J_+$, for $N\in\mathbb{N}^*$ and $0\leq\ell\leq 2^N-1$, satisfy the following properties, some of which are illustrated on Figure 1 below:

(1) For any $N\in\mathbb{N}^*$ and any $0\leq\ell\leq 2^N-1$, C^N_ℓ is a positive cone with vertex at V, i.e.,

$$V+\lambda(W-V)\in C^N_\ell,\quad\forall\lambda>0,\ \forall W\in C^N_\ell;$$

(2) For any $N\in\mathbb{N}^*$, $(C^N_\ell)_{0\leq\ell\leq 2^N-1}$ is a partition of $V+\mathbb{R}^J_+$, i.e.,

$$\bigcup_{\ell=0}^{2^N-1}C^N_\ell=V+\mathbb{R}^J_+\quad\text{and}\quad C^N_\ell\cap C^N_{\ell'}=\emptyset,\ \forall\ell\neq\ell';$$

(3) Each partition is a refinement of the preceding one: for any $N \geq 2$ and any $0 \leq \ell \leq 2^N - 1$, there exists a unique $0 \leq \ell' \leq 2^{N-1} - 1$ such that $C_\ell^N \subset C_{\ell'}^{N-1}$.

Denote by $\mathscr{L}$ the set of half-lines issued from V and contained in $V + \mathbb{R}_+^J$. Note that $\mathscr{L}$ is parametrized by $s \in \mathbf{S}_{\mathcal{J}}$. We also assume the following property:

(4) For any $L \in \mathscr{L}$ parametrized by $s \in \mathbf{S}_{\mathcal{J}}$, there exists a subsequence $(C_{\ell(s,N)}^N)_{N \in \mathbb{N}^*}$ which converges to $\mathscr{L}$, in the following sense. There exists $d : \mathbb{N} \to \mathbb{R}^+$ with $d \to 0$ as $N \to +\infty$, such that, for any $s' \in \mathbf{S}_{\mathcal{J}}$ parametrizing a half-line $L' \in \mathscr{L}$ contained in $\mathbf{S}_{\ell(s,N)}^N$, we have

$$\|s' - s\|_1 \leq d(N). \tag{21}$$

This last property is equivalent to saying that the size of the positive cones tends uniformly to 0 as $N \to +\infty$.

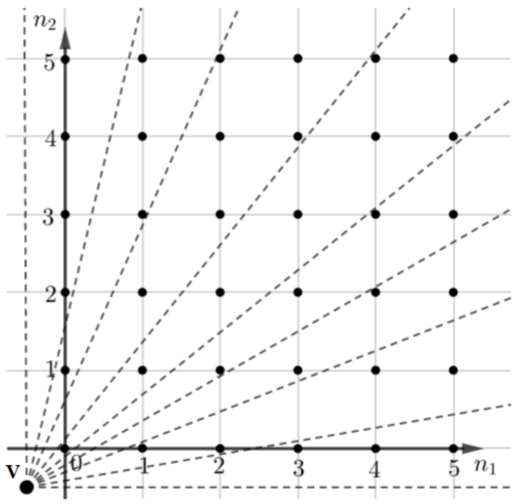

Figure 1: The positive cones C_ℓ^N, for $J = 2$, $N = 3$.

Remark 17. *The positive cones C_ℓ^N can be seen as positive sub-cones of the Heisenberg fan (whose definition is recalled in Section 1.7).*

Spectral decomposition. Decomposing φ_k on the spaces $\mathcal{H}_{(n_j)}^{\mathcal{J}}$ defined in Section 3.2, we write

$$\varphi_k = \sum_{\ell=0}^{2^N-1} \varphi_{k,\ell}^N \tag{22}$$

where

$$\varphi_{k,\ell}^N = \sum_{(n_1,\dots,n_J) \in C_\ell^N} \varphi_{k,n_1,\dots,n_J}$$

and, for any $(n_j) \in \mathbb{N}^{\mathcal{J}}$, $k \in \mathbb{N}^*$ and $j \in \mathcal{J}$,

$$\Omega_j \varphi_{k,n_1,\dots,n_J} = (2n_j + 1)\varphi_{k,n_1,\dots,n_J}.$$

For any $N \in \mathbb{N}^*$ and any $0 \leq \ell \leq 2^N - 1$, we take ν_ℓ^N to be a microlocal defect measure of the sequence $(\varphi_{k,\ell}^N)_{k \in \mathbb{N}^*}$. By diagonal extraction in $k \in \mathbb{N}^*$ (which we omit in the notations), we can assume that any of these microlocal defect measures is obtained with respect to the same subsequence.

Lemma 18. *The following properties hold:*

(1) All the mass of ν_ℓ^N is contained in $S\Sigma_{\mathcal{J}}$ for any $N \in \mathbb{N}^*$ and any $0 \leq \ell \leq 2^N - 1$;

(2) For $N \in \mathbb{N}^*$ and $\ell \neq \ell'$ with $0 \leq \ell, \ell' \leq 2^N - 1$, the joint microlocal defect measure (see Definition 5) of $(\varphi_{k,\ell}^N)_{k\in\mathbb{N}^*}$ and $(\varphi_{k,\ell'}^N)_{k\in\mathbb{N}^*}$ vanishes. In particular, for any $N \in \mathbb{N}^*$,

$$\nu = \sum_{\ell=0}^{2^N-1} \nu_\ell^N. \tag{23}$$

Proof. The proof mainly relies on averaging techniques (see also Appendix C for a result obtained by these techniques in the much simpler context of flat contact sub-Laplacians).

We first prove Point (1). Using (17), (22) and the fact that $P_n^{\mathcal{J}} \in \Psi^0(\mathbf{H}^m)$ commutes with the operators Ω_j and R_j, we get that

$$\varphi_{k,\ell}^N = P_{r(k)}^{\mathcal{J}} \varphi_{k,\ell}^N.$$

Point (1) now follows from the fact that $\sigma_P(P_{r(k)}^{\mathcal{J}}) \to \mathbf{1}_{\Sigma_{\mathcal{J}}}$ as $k \to +\infty$ (see the proof of Theorem 1).

We now turn to the proof of Point (2). Let N, ℓ, ℓ' be as in the statement. By Point (1) and Lemma 14, we know that the joint microlocal defect measure of $(\varphi_{k,\ell}^N)_{k\in\mathbb{N}^*}$ and $(\varphi_{k,\ell'}^N)_{k\in\mathbb{N}^*}$ has no mass outside $S\Sigma_{\mathcal{J}}$.

Let $b \in \mathscr{S}^0(\mathbf{H}^m)$ which is microlocally supported in a conic set in which R_j, Ω_j act as first-order pseudodifferential operators for any $j \in \mathcal{J}$. A typical example of microlocal support for b is given by any conic subset of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j} = 0\}$ is reduced to 0, for any $j \in \mathcal{J}$. We set $U(t) = U(t_1, \ldots, t_J) = e^{i(t_1\Omega_1 + \ldots + t_J\Omega_J)}$ for $t = (t_1, \ldots, t_J) \in (\mathbb{R}/2\pi\mathbb{Z})^J$.

The average of $\mathrm{Op}(b)$ is then defined by

$$A = \int_{(\mathbb{R}/2\pi\mathbb{Z})^J} U(-t)\mathrm{Op}(b)U(t)dt$$

(see [Wei77]). For $1 \leq j \leq J$, since

$$\frac{d}{dt_j} U(-t)\mathrm{Op}(b)U(t) = U(-t)[\mathrm{Op}(b), \Omega_j]U(t),$$

integrating in the t_j variable, using that all Ω_i commute together, and that $\exp(2i\pi\Omega_j) = \mathrm{Id}$ (since the eigenvalues of Ω_j belong to $\mathbb{N}$), we get that $[A, \Omega_j] = 0$ for any $1 \leq j \leq J$.

By a bracket computation, A has principal symbol

$$a := \sigma_P(A) = \int_{(\mathbb{R}/2\pi\mathbb{Z})^J} b \circ \theta_1(t_1) \circ \ldots \circ \theta_J(t_J)\ dt.$$

Here, $\theta_j(\cdot)$ denotes, for $1 \leq j \leq J$, the 2π-periodic flow of the Hamiltonian vector field of $\sigma_P(\Omega_j)$ (see [CdVHT18, Lemma 6.1] for similar arguments).

Remark 19. *If D is a 0^{th}-order pseudodifferential operator on $\mathbf{H}^m$ which satisfies $[D, \Omega_j] = 0$ for any $j \in \mathcal{J}$, then D leaves $\mathcal{H}_{(n_j)}^{\mathcal{J}}$ invariant for any $(n_j) = (n_1, \ldots, n_J) \in \mathbb{N}$. It follows that for any $f \in \mathcal{H}_{(n_j)}^{\mathcal{J}}$ and any $g \in \mathcal{H}_{(n_j')}^{\mathcal{J}}$ such that $(n_1, \ldots, n_J) \neq (n_1', \ldots, n_J')$, we have $(Df, g) = 0$.*

We know that $\sigma_P(A) = b$ on $S\Sigma_{\mathcal{J}}$. Therefore,

$$(\mathrm{Op}(b)\varphi_{k,\ell}^N, \varphi_{k,\ell'}^N) - (A\varphi_{k,\ell}^N, \varphi_{k,\ell'}^N) \underset{k\to+\infty}{\longrightarrow} 0.$$

Since A commutes with Ω_j for any $1 \leq j \leq J$, by Remark 19, we know that $(A\varphi_{k,\ell}^N, \varphi_{k,\ell'}^N) = 0$. Hence, $(\mathrm{Op}(b)\varphi_{k,\ell}^N, \varphi_{k,\ell'}^N)$ tends to 0 as $k \to +\infty$. Using this result for all possible b with microlocal support satisfying the property recalled at the beginning of the proof, we obtain that the joint microlocal defect measure of $(\varphi_{k,\ell}^N)_{k\in\mathbb{N}^*}$ and of $(\varphi_{k,\ell'}^N)_{k\in\mathbb{N}^*}$ vanishes. Evaluating $(\mathrm{Op}(b)\varphi_k, \varphi_k)$ in the limit $k \to +\infty$ and using (22), we conclude the proof of Point (2). $\square$

Approximate invariance. We fix $N \in \mathbb{N}^*$ and $0 \leq \ell \leq 2^N - 1$ and we consider $s \in \mathbf{S}_{\mathcal{J}}$ such that the half-line issued from V and defined by the J equations $\frac{2x_j+1}{2x_1+1+\ldots+2x_J+1} = s_j$ (and $x_j \geq -1/2$) lies in C_ℓ^N.

Let A be a 0-th order pseudodifferential operator microlocally supported in a conic set where R_j, Ω_j act as first-order pseudodifferential operators for any $j \in \mathcal{J}$. Assume moreover that A commutes with $\Omega_1, \ldots, \Omega_J$ and with $\partial_{x_j}, \partial_{y_j}$ and ∂_{z_j} for any $J+1 \leq j \leq m$. Recall that R_s was defined in (11). Using that $[A, R_s]$ commutes with $\Omega_1, \ldots, \Omega_J$ in order to kill crossed terms (see Remark 19), we have

$$\begin{aligned}
([A, R_s]\varphi_{k,\ell}^N, \varphi_{k,\ell}^N) &= ([A, R_s] \sum_{(n_1,\ldots,n_J)\in C_\ell^N} \varphi_{k,n_1,\ldots,n_J}, \sum_{(n_1,\ldots,n_J)\in C_\ell^N} \varphi_{k,n_1,\ldots,n_J}) \\
&= \sum_{(n_1,\ldots,n_J)\in C_\ell^N} ([A, R_s]\varphi_{k,n_1,\ldots,n_J}, \varphi_{k,n_1,\ldots,n_J}) \qquad (24)
\end{aligned}$$

Let us fix $(n_1, \ldots, n_J) \in C_\ell^N$ and prove that

$$\begin{aligned}
&([A, R_s]\varphi_{k,n_1,\ldots,n_J}, \varphi_{k,n_1,\ldots,n_J}) \\
&\qquad = \sum_{j=1}^{J} \left(s_j - \frac{2n_j + 1}{\sum_{i=1}^J 2n_i + 1} \right) ([A, R_j]\varphi_{k,n_1,\ldots,n_J}, \varphi_{k,n_1,\ldots,n_J}) \qquad (25)
\end{aligned}$$

We set

$$R = \frac{\sum_{j=1}^J (2n_j + 1)R_j - \sum_{i=J+1}^m \Delta_i}{\sum_{j=1}^J 2n_j + 1}.$$

and, for the sake of simplicity of notations, $\varphi = \varphi_{k,n_1,\ldots,n_J}$. Using that R is selfadjoint (since R_j is selfadjoint for any j) and that φ is an eigenfunction of R, we get

$$([A, R]\varphi, \varphi) = (AR\varphi, \varphi) - (A\varphi, R\varphi) = 0$$

and therefore, since A commutes with $\Delta_{J+1}, \ldots, \Delta_m$, we get

$$([A, R_s]\varphi, \varphi) = ([A, R_s - R]\varphi, \varphi) = \sum_{j=1}^{J} \left(s_j - \frac{2n_j + 1}{\sum_{i=1}^J 2n_i + 1} \right) ([A, R_j]\varphi, \varphi)$$

which is exactly (25).

Thanks to our choice of microlocal support for A, we know that $[A, R_j] \in \Psi^0(\mathbf{H}^m)$ for $1 \leq j \leq J$. Combining (24) and (25), we obtain

$$\begin{aligned}
\left|([A, R_s]\varphi_{k,\ell}^N, \varphi_{k,\ell}^N)\right| &\leq C \sum_{(n_1,\ldots,n_J)\in C_\ell^N} \sum_{j=1}^{J} \left| s_j - \frac{2n_j + 1}{\sum_{i=1}^J 2n_i + 1} \right| \|\varphi_{k,n_1,\ldots,n_J}\|_{L^2}^2 \\
&\leq Cd(N) \|\varphi_{k,\ell}^N\|_{L^2}^2
\end{aligned} \qquad (26)$$

where in the last line, we used (21) and the fact that the decomposition (20) is orthogonal.

In order to pass to the limit $k \to +\infty$ in these last inequalities, we note that

$$\sigma_P([A, R_s])_{|\Sigma_{\mathcal{J}}} = \{a_{|\Sigma_{\mathcal{J}}}, \rho_s\}_{\omega_{|\Sigma_{\mathcal{J}}}} \qquad (27)$$

(see [CdVHT18, Lemma 6.2] for a similar identity). Here, the Poisson bracket $\{\cdot, \cdot\}_{\omega_{|\Sigma_{\mathcal{J}}}}$ is the Poisson bracket on the manifold $(\Sigma_{\mathcal{J}}, \omega_{|\Sigma_{\mathcal{J}}})$ which is symplectic as it is defined as a product of symplectic manifolds (recall that for $m = 1$, the 4-dimensional manifold Σ is symplectic, see for example [CdVHT18]).

Since all the mass of ν_ℓ^N is contained in $S\Sigma_{\mathcal{J}}$ by Lemma 18, we finally deduce from (26) the upper bound

$$\int_{S\Sigma_{\mathcal{J}}} \{a_{|\Sigma_{\mathcal{J}}}, \rho_s\}_{\omega_{|\Sigma_{\mathcal{J}}}} \, d\nu_\ell^N \leq Cd(N)\nu_\ell^N(S\Sigma_{\mathcal{J}}). \qquad (28)$$

The upper bound (28) has been established only for $a_{|\Sigma_{\mathcal{J}}}$ the restriction to $\Sigma_{\mathcal{J}}$ of the symbol of an operator A of order 0 *which commutes with* $\Omega_1, \ldots, \Omega_J$ *and* $\partial_{x_j}, \partial_{y_j}$ *and* ∂_{z_j} *for any* $J+1 \leq j \leq m$, and we would like to remove this commutation assumption. Let $b \in \mathscr{S}^0(\mathbf{H})$ of the form

$$b(q,p) = b_{\mathcal{J}}(q_1, \ldots, q_J, p_1, \ldots, p_J)$$

where (q,p) denote the coordinates in $T^*\mathbf{H}^m$, (q_j, p_j) the coordinates in the cotangent bundle of the j-th copy of $\mathbf{H}$, and $b_{\mathcal{J}} \in \mathscr{S}^0(\mathbf{H}^{\mathcal{J}})$ is an arbitrary 0-th order symbol supported in a subset of $T^*\mathbf{H}^{\mathcal{J}}$ where R_j, Ω_j act as first-order pseudodifferential operators for any $j \in \mathcal{J}$. We consider the operator

$$A = \int_{(\mathbb{R}/2\pi\mathbb{Z})^J} U(-t)\mathrm{Op}(b)U(t)dt \ \in \Psi^0(\mathbf{H}^m)$$

where $U(t) = U(t_1, \ldots, t_J) = e^{i(t_1\Omega_1 + \ldots + t_J\Omega_J)}$ for $t = (t_1, \ldots, t_J) \in (\mathbb{R}/2\pi\mathbb{Z})^J$. By an argument that we have already in the proof of Point (2) of Lemma (18), A commutes with Ω_j for any $1 \leq j \leq J$, and it also commutes with $\partial_{x_j}, \partial_{y_j}$ and ∂_{z_j} for any $J+1 \leq j \leq m$. Moreover, the principal symbol of A on $S\Sigma_{\mathcal{J}}$ coincides with $b_{\mathcal{J}}$ by the Egorov theorem. Using (28) for A, this proves that (28) is valid for any symbol a of order 0 on $\mathbf{H}^m$ supported far from the sets $\{p_{z_j} = 0\}$ for $j \in \mathcal{J}$, without any assumption of commutation on A.

Disintegration of measures. From the equality (23) taken in the limit $N \to +\infty$, we will deduce that $\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu_s^{\mathcal{J}} dQ^{\mathcal{J}}(s)$. Note that a simple Fubini argument does not suffice since $Q^{\mathcal{J}}$ is not the Lebesgue measure in general (it may contain Dirac masses, see Section 1.6). Instead, we have to adapt the proof of the classical disintegration of measure theorem (see [Roh62]).

First of all, we define a measure $Q^{\mathcal{J}}$ over $\mathbf{S}_{\mathcal{J}}$ as follows. It was explained at the beginning of Section 3.3 that the set $\mathscr{L}$ of half-lines issued from V and contained in $V + \mathbb{R}_+^J$ is parametrized by $s \in \mathbf{S}_{\mathcal{J}}$. For $N \in \mathbb{N}^*$ and $0 \leq \ell \leq 2^N - 1$, we consider the subset of $\mathbf{S}_{\mathcal{J}}$ given by

$$\mathbf{S}_\ell^N = \left\{ s \in \mathbf{S}_{\mathcal{J}},\ s \text{ parametrizes a half-line of } \mathscr{L} \text{ contained in } C_\ell^N \right\}. \tag{29}$$

Then we define

$$Q^{\mathcal{J}}(\mathbf{S}_\ell^N) = \nu_\ell^N(S\Sigma). \tag{30}$$

This definition is consistent thanks to the partition of $V + \mathbb{R}_+^J$ into nested positive cones: $Q^{\mathcal{J}}$ is well-defined on any $\mathbf{S}_\ell^N$ and it is also additive. By the properties of the positive cones C_ℓ^N, for any $s \in \mathbf{S}_{\mathcal{J}}$, there exists a sequence $(\ell(s,N))_{N \in \mathbb{N}^*}$ such that $\mathbf{S}_{\ell(s,N)}^N \subset \mathbf{S}_{\mathcal{J}}$ converges to s, in the sense that any sequence $(s^N)_{N \in \mathbb{N}^*}$ such that $s^N \in \mathbf{S}_{\ell(s,N)}^N$ for any $N \in \mathbb{N}^*$ converges to s as $N \to +\infty$. Therefore, by extension, (30) defines a (unique) non-negative Radon measure $Q^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$.

Given $N \geq 1$, $0 \leq \ell \leq 2^N - 1$ and a continuous function $f : S\Sigma_{\mathcal{J}} \to \mathbb{R}$, we set

$$f_\ell^N = \frac{1}{\nu_\ell^N(S\Sigma_{\mathcal{J}})} \int_{S\Sigma_{\mathcal{J}}} f d\nu_\ell^N \tag{31}$$

if $\nu_\ell^N(S\Sigma_{\mathcal{J}}) \neq 0$, and $f_\ell^N = 0$ otherwise.

Proposition 20. *Given any continuous function* $f : S\Sigma \to \mathbb{R}$, *for* $Q^{\mathcal{J}}$*-almost all* $s \in \mathbf{S}_{\mathcal{J}}$, *there exists a real number* $e(f)(s)$ *such that*

$$f_{\ell(s,N)}^N \underset{N \to +\infty}{\longrightarrow} e(f)(s),$$

where, for any $N \in \mathbb{N}^*$, $\ell(s,N)$ *is the unique integer* $0 \leq \ell(s,N) \leq 2^N - 1$ *such that* $s \in \mathbf{S}_{\ell(s,N)}^N$.
In the sequel, we call $\ell(s,N)$ *the approximation at order* N *of* s.

Proof. By linearity of formula (31), it is sufficient to prove the statement for $f \geq 0$. Therefore, in the sequel, we fix $f \geq 0$. For $N \geq 1$, we define the function $f^N : \mathbf{S}_{\mathcal{J}} \to \mathbb{R}$ by $f^N(s) = f^N_{\ell(s,N)}$, where $\ell(s,N)$ is the approximation at order N of s. Note that f^N is constant on $\mathbf{S}^N_\ell$ for $0 \leq \ell \leq 2^N - 1$.

For $0 \leq \alpha < \beta \leq 1$, we define $S(\alpha,\beta)$ as the set of $s \in \mathbf{S}_{\mathcal{J}}$ such that

$$\liminf_{N\to+\infty} f^N(s) < \alpha < \beta < \limsup_{N\to+\infty} f^N(s).$$

To prove Proposition 20, it is sufficient to prove that $S(\alpha,\beta)$ has $Q^{\mathcal{J}}$-measure 0 for any $0 \leq \alpha < \beta \leq 1$. Fix such α, β. For $s \in S(\alpha,\beta)$, take a sequence $1 \leq N^\alpha_1(s) < N^\beta_1(s) < N^\alpha_2(s) < N^\beta_2(s) < \ldots < N^\alpha_k(s) < N^\beta_k(s) < \ldots$ of integers such that $f^{N^\alpha_k(s)}(s) < \alpha$ and $f^{N^\beta_k(s)}(s) > \beta$ for any $k \geq 1$. We finally define the following sets:

$$A_k = \bigcup_{s\in S(\alpha,\beta)} \mathbf{S}^{N^\alpha_k(s)}_{\ell(s,N^\alpha_k(s))}$$

$$B_k = \bigcup_{s\in S(\alpha,\beta)} \mathbf{S}^{N^\beta_k(s)}_{\ell(s,N^\beta_k(s))}$$

We have $S(\alpha,\beta) \subset A_{k+1} \subset B_k \subset A_k$ for every $k \geq 1$. In particular,

$$S(\alpha,\beta) \subset \widetilde{S}(\alpha,\beta) := \bigcap_{k\in\mathbb{N}^*} A_k = \bigcap_{k\in\mathbb{N}^*} B_k. \tag{32}$$

Given any two of the sets $\mathbf{S}^{N^\alpha_k(s)}_{\ell(s,N^\alpha_k(s))}$ that form A_k, either they are disjoint or one is contained in the other. Consequently, A_k may be written as a disjoint union of such sets, denoted by $A^{k'}_k$. Therefore,

$$\int_{A_k} f dQ^{\mathcal{J}} = \sum_{k'} \int_{A^{k'}_k} f dQ^{\mathcal{J}} < \sum_{k'} \alpha Q^{\mathcal{J}}(A^{k'}_k) = \alpha Q^{\mathcal{J}}(A_k)$$

and analogously, with similar notations,

$$\int_{B_k} f dQ^{\mathcal{J}} = \sum_{k'} \int_{B^{k'}_k} f dQ^{\mathcal{J}} > \sum_{k'} \beta Q^{\mathcal{J}}(B^{k'}_k) = \beta Q^{\mathcal{J}}(B_k).$$

Since $B_k \subset A_k$, we get $\alpha Q^{\mathcal{J}}(A_k) > \beta Q^{\mathcal{J}}(B_k)$. Taking the limit $k \to +\infty$, it yields $\alpha Q^{\mathcal{J}}(\widetilde{S}(\alpha,\beta)) > \beta Q^{\mathcal{J}}(\widetilde{S}(\alpha,\beta))$, which is possible only if $Q^{\mathcal{J}}(\widetilde{S}) = 0$. Therefore, using (32), we get $Q^{\mathcal{J}}(S) = 0$, which concludes the proof of the proposition. $\square$

From (23) and (31), we infer that for any $N \geq 1$,

$$\int_{S\Sigma_{\mathcal{J}}} f d\nu^{\mathcal{J}} = \sum_{\ell=0}^{2^N-1} \int_{S\Sigma_{\mathcal{J}}} f d\nu^N_\ell = \sum_{\ell=0}^{2^N-1} f^N_\ell \nu^N_\ell(S\Sigma_{\mathcal{J}}),$$

and the dominated convergence theorem together with the definition of $Q^{\mathcal{J}}$ and Proposition 20 yield

$$\int_{S\Sigma_{\mathcal{J}}} f d\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} e(f)(s) dQ^{\mathcal{J}}(s). \tag{33}$$

We see that for a fixed $s \in \mathbf{S}_{\mathcal{J}}$,

$$C^0(S\Sigma_{\mathcal{J}}, \mathbb{R}) \ni f \mapsto e(f)(s) \in \mathbb{R}$$

is a non-negative linear functional on $C^0(S\Sigma_{\mathcal{J}}, \mathbb{R})$. By the Riesz-Markov theorem, there exists a unique Radon probability measure $\nu^{\mathcal{J}}_s$ on $S\Sigma_{\mathcal{J}}$ such that

$$e(f)(s) = \int_{S\Sigma_{\mathcal{J}}} f d\nu^{\mathcal{J}}_s. \tag{34}$$

Putting (33) and (34) together, we get

$$\int_{S\Sigma_{\mathcal{J}}} f d\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \left(\int_{S\Sigma_{\mathcal{J}}} f d\nu_s^{\mathcal{J}} \right) dQ^{\mathcal{J}}(s)$$

which is the desired disintegration of measures formula.

Conclusion of the proof. There remains to show that $\nu_s^{\mathcal{J}}$ is invariant by $\vec{\rho}_s^{\mathcal{J}}$. Let $a \in \mathscr{S}^0(\mathbf{H}^m)$ be supported in cone of $T^*\mathbf{H}^m$ whose intersection with some conic neighborhood of the set $\{p_{z_j} = 0\}$ is reduced to 0, for any $j \in \mathcal{J}$. For $Q^{\mathcal{J}}$-almost every $s \in \mathbf{S}_{\mathcal{J}}$, we have

$$\begin{aligned}
\int_{S\Sigma_{\mathcal{J}}} \{a, \rho_s^{\mathcal{J}}\} d\nu_s^{\mathcal{J}} &= e(\{a, \rho_s^{\mathcal{J}}\})(s) \qquad \text{(by (34))} \\
&= \lim_{N\to+\infty} \frac{1}{\nu_{\ell(s,N)}^N(S\Sigma_{\mathcal{J}})} \int_{S\Sigma_{\mathcal{J}}} \{a, \rho_s^{\mathcal{J}}\} d\nu_{\ell(s,N)}^N \qquad (35) \\
&\leq \lim_{N\to+\infty} Cd(N) \qquad \text{(by (28))} \\
&= 0
\end{aligned}$$

with the convention that if the denominator in (35) is null, then the whole expression is null. For an arbitrary $a \in \mathscr{S}^0(\mathbf{H}^m)$, taking a sequence $a_n \in \mathscr{S}^0(\mathbf{H}^m)$ whose support has the above property and such that $a_n \to a$ in $S\Sigma_{\mathcal{J}}$ (in the space of symbols) as $n \to +\infty$, we see that the above quantity also vanishes since $\nu_s^{\mathcal{J}}$ has finite mass and $\{a_n, \rho_s^{\mathcal{J}}\} \to \{a, \rho_s^{\mathcal{J}}\}$ in $S\Sigma_{\mathcal{J}}$ as $n \to +\infty$. This implies that $\nu_s^{\mathcal{J}}$ is invariant by the flow $e^{t\vec{\rho}_s^{\mathcal{J}}}$, which concludes the proof of Theorem 2.

4 Proof of Theorem 3

In this section, we prove Theorem 3. The four steps are the following:

1. In Lemma 21 and Lemma 22, we prove the result for a fixed $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, $Q^{\mathcal{J}}$ the Dirac mass at some $s \in \mathbf{S}_{\mathcal{J}}$, and $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$
   (i) has no mass outside $S\Sigma_{\mathcal{J}}$,
   (ii) is invariant under the flow of $\vec{\rho}_s^{\mathcal{J}}$,
   (iii) and *is in a simple tensor form* that we make precise below.

   In other words, if $\nu_\infty = \nu_s^{\mathcal{J}}$ with $\nu_s^{\mathcal{J}}$ satisfying (i), (ii) and (iii), then it is a QL.
2. In Lemma 24, we extend the result of Step 1 to the case where (iii) is not necessarily satisfied, i.e., $\nu_\infty = \nu_s^{\mathcal{J}}$ satisfies only (i) and (ii).
3. In Lemma 26, we extend the result of Steps 1 and 2 to the case where $\nu_\infty \in \mathscr{P}_{S\Sigma}$ has no mass outside $S\Sigma_{\mathcal{J}}$ for some $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, i.e., $\nu_\infty = \nu^{\mathcal{J}}$.
4. Finally, using the previous result for all $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, we prove Theorem 3 in full generality (i.e., for arbitrary $\nu_\infty \in \mathscr{P}_{S\Sigma}$).

The specific algebraic structure of $\mathrm{sp}(-\Delta)$ plays a key role at each of these four steps. Note that similar roadmaps have been followed in different but related contexts, see [JZ96] and [Stu19].

The map $\Sigma \to \mathbf{H}^m \times \mathbb{R}^m$, $(q, p) \mapsto (q, p_{z_1}, \ldots, p_{z_m})$ is an isomorphism, and thus, in the sequel, we consider the coordinates $(q, p_{z_1}, \ldots, p_{z_m})$ on Σ and the coordinates $(q, p_{z_1} : \cdots : p_{z_m})$ on $S\Sigma$, where the notation $p_{z_1} : \cdots : p_{z_m}$ stands for homogeneous coordinates.

Let us summarize the proof, which uses in a key way the precise description of the spectrum of $-\Delta$ (see Section 1.6) and the knowledge of the flows of the Hamiltonian vector fields $\vec{\rho}_s^{\mathcal{J}}$ (see Remark 7).

We fix $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. Since any two of the operators R_j and $\Omega_{j'}$ for $j, j' \in \mathcal{J}$ commute, the orthogonal decomposition (20) can be refined: more precisely, given $(n_j) \in \mathbb{N}^{\mathcal{J}}$ and $(\alpha_j) \in (\mathbb{Z} \setminus \{0\})^{\mathcal{J}}$, we consider the joint eigenspace $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)} \subset L^2(\mathbf{H}^m)$ on which the operator $\frac{1}{i}\partial_{z_j}$ acts as α_j and Ω_j acts as $2n_j + 1$.

ν_∞ is obtained as a QL of a sequence of normalized eigenfunctions $(\varphi_k)_{k \in \mathbb{N}^*}$ which is described through its components in these eigenspaces. Moreover, each of the four steps is achieved by taking linear combinations of eigenfunctions (with same eigenvalues) used in the previous step. Therefore, the number of eigenspaces $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ used for building $(\varphi_k)_{k \in \mathbb{N}^*}$ increases at each step.

In order to achieve Step 1, we focus on the eigenspaces $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ corrreponding to

$$\frac{2n_j + 1}{\sum_{i \in \mathcal{J}}(2n_i + 1)} \approx s_j \quad \text{and} \quad \frac{\alpha_j}{\alpha_{j'}} \approx \frac{p_{z_j}}{p_{z_{j'}}}$$

for any $j, j' \in \mathcal{J}$.

For Step 2, we add the results of the previous step for different $p \in S\Sigma_{\mathcal{J}}$, and we take care that each term in the sum corresponds to the same value of $-\Delta$. Hence, $(n_j) \in \mathbb{N}^{\mathcal{J}}$ is the same as in Step 1, but we use various $(\alpha_j) \in (\mathbb{Z} \setminus \{0\})^{\mathcal{J}}$ to reach all p.

For Step 3, we add the results of Step 2 for different $s \in \mathbf{S}_{\mathcal{J}}$. Therefore, we use the eigenspaces $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ also for different $(n_j) \in \mathbb{N}^{\mathcal{J}}$. Finally, in Step 4, we sum the sequences obtained at Step 3 for $\mathcal{J}$ ranging over $\mathcal{P} \setminus \{0\}$.

In order to describe the measures in a "tensor form" which we consider for Step 1, we need to introduce a few notations.

Notations. For the first three steps, we fix $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$. Any $s \in \mathbf{S}_{\mathcal{J}}$ can be identified to some homogeneous coordinate $p_{z_1} : \cdots : p_{z_m}$ (with $p_{z_i} = 0$ for $i \notin \mathcal{J}$), in a way which does not depend on $q \in \mathbf{H}^m$. Thus, for any $q \in \mathbf{H}^m$, $t \in \mathbb{R}$ and $s \in \mathbf{S}_{\mathcal{J}}$, it makes sense to consider the point $q + ts \in \mathbf{H}^m$, which has the same coordinates x_j and y_j as q for any $1 \leq j \leq m$ (only the coordinates z_j for $j \in \mathcal{J}$ change).

Let us consider the set

$$M_q^s = \overline{\{q + ts,\ t \in \mathbb{R}\}} \subset \mathbf{H}^m$$

where the bar denotes the closure in $\mathbf{H}^m$. The set M_q^s is a submanifold of $\mathbf{H}^m$ of dimension $d_q^s \leq m$, and we denote by $\mathscr{H}_q^s$ the Hausdorff measure of dimension d_q^s on M_q^s.

For any $(q, p) \in S\Sigma$ and any $q' \in \mathbf{H}^m$, it makes sense to consider the point $(q', p) \in S\Sigma$, which is the point in the fiber of $S\Sigma$ over q that has the same homogeneous coordinates $p_{z_1} : \cdots : p_{z_m}$ as p.

Lemma 21. *Let $(q, p) \in S\Sigma_{\mathcal{J}}$ and $s \in \mathbf{S}_{\mathcal{J}}$ be such that there exists a J-tuple $(n_j) \in \mathbb{N}^{\mathcal{J}}$ with*

$$s_j = \frac{2n_j + 1}{\sum\limits_{i \in \mathcal{J}} (2n_i + 1)} \tag{36}$$

for any $j \in \mathcal{J}$. Then, the measure $\mathscr{H}_q^s \otimes \delta_p$ is a Quantum Limit. [The associated sequence of normalized eigenfunctions is specified in the proof, see also Remark 23.]

Proof. Since the s_j are pairwise rationally related, the mapping $t \mapsto q + ts$ is periodic and $d_q^s = 1$. Without loss of generality, we assume that $\mathcal{J} = \{1, \ldots, J\}$ for some $1 \leq J \leq m$.

We construct a sequence of eigenfunctions $(\varphi_k)_{k \in \mathbb{N}^*}$ of $-\Delta$ which admits $\mu_{q,p}^s$ as unique Quantum Limit. In our construction, for any $k \in \mathbb{N}^*$, φ_k belongs to the eigenspace $\mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}$ for some $(n_j) \in \mathbb{N}^{\mathcal{J}}$ and some $(\alpha_j) \in (\mathbb{Z} \setminus \{0\})^{\mathcal{J}}$, and it does not depend on the variables in the i-th copy of $\mathbf{H}$ for $i \notin J$. Our goal is to choose adequately the J-tuples (n_j) and (α_j). Note that a similar argument for $m = 1$ is done in the proof of Point 2 of Proposition 3.2 in [CdVHT18].

We fix a sequence of J-tuples $(\alpha_{1,k}, \ldots, \alpha_{J,k}) \in (\mathbb{Z} \setminus \{0\})^J$, for $k \in \mathbb{N}^*$, such that:

- For any $1 \leq j \leq J$, $\alpha_{j,k} \to +\infty$ as $k \to +\infty$, so that for any $1 \leq j, j' \leq J$, there holds

$$\frac{n_{j'}}{\alpha_{j,k}} \underset{k\to+\infty}{\longrightarrow} 0; \tag{37}$$

- For any $1 \leq j, j' \leq J$,

$$\frac{\alpha_{j,k}}{\alpha_{j',k}} \underset{k\to+\infty}{\longrightarrow} \frac{p_{z_j}}{p_{z_{j'}}}, \tag{38}$$

  where $p_{z_1} : \dots : p_{z_m}$ are the homogeneous coordinates of p in $S\Sigma$.

Now, for any $k \in \mathbb{N}^*$, denoting by $\mathbf{1}$ the constant function equal to 1 (on some copy of $\mathbf{H}$), we define

$$\varphi_k = \Phi_k^1 \otimes \ldots \otimes \Phi_k^J \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{m-J \text{ times}}, \tag{39}$$

where, for $1 \leq j \leq J$,

$$\Phi_k^j(x_j, y_j, z_j) = \phi_{j,k}(x_j, y_j)e^{i\alpha_{j,k}z_j}$$

is an eigenfunction of $-\Delta_j$ (on the j-th copy of $\mathbf{H}$) with eigenvalue $(2n_j + 1)|\alpha_{j,k}|$. The precise form of $\phi_{j,k}$ will be given below.

Using (37) and the proof of Theorem 1, notably the pseudodifferential operators $P_n^{\mathcal{J}}$ introduced in (14), we obtain that the mass of any Quantum Limit of $(\varphi_k)_{k\in\mathbb{N}^*}$ is contained in $S\Sigma_{\mathcal{J}}$. Moreover, from the decomposition into cones done in Section 3.3 and the equality (36), we infer that any Quantum Limit of $(\varphi_k)_{k\in\mathbb{N}^*}$ is invariant under $\vec{\rho}_s^{\mathcal{J}}$.

In the next paragraphs, we explain how to choose $\phi_{j,k}$ with eigenvalue $2n_j + 1$ in order to ensure that $(\varphi_k)_{k\in\mathbb{N}^*}$ has a unique QL, which is $\mu_{0,p}^s$. For the sake of simplicity of notations, we set $\alpha = \alpha_{j,k}$. The eigenspace of $-\Delta_j$ corresponding to the eigenvalue $(2n_j + 1)|\alpha|$ is of the form $(A_\alpha^*)^{n_j}(\ker(A_\alpha))e^{i\alpha z}$, where $A_\alpha = \partial_{x_j} + i\partial_{y_j} + i\alpha x_j$ locally, and, accordingly, $A_\alpha^* = -\partial_{x_j} + i\partial_{y_j} + i\alpha x_j$ locally (see for example [CdV84, Section 2]). This follows from a Fourier expansion in the z_j variable, which gives

$$-\Delta_j = \bigoplus_{\gamma\in\mathbb{Z}} B_\gamma, \qquad \text{where } \ B_\gamma = A_\gamma^* A_\gamma + \gamma \text{ for } \gamma \in \mathbb{Z}.$$

We note that the function $f_{j,k}(x_j, y_j) = c_k \exp(-\alpha\frac{x_j^2}{2} + \frac{\alpha}{4}(x_j + iy_j)^2)$ (normalized to 1 thanks to c_k) is a quasimode of A_α, as $\alpha \to +\infty$, for the eigenvalue 0. Moreover, a well-known computation on coherent states (see Example 1 of Chapter 5 in [Zwo12]) guarantees that for any $a \in \mathscr{S}^0(\mathbb{R}^{2m})$,

$$(\mathrm{Op}(a)(A_\alpha^*)^{n_j} f_{j,k}, (A_\alpha^*)^{n_j} f_{j,k}) \underset{k\to+\infty}{\longrightarrow} a(0, 0).$$

In other words, $(A_\alpha^*)^{n_j} f_{j,k}$, seen as a sequence of functions of $\mathbb{R}^{2m}$, has a unique Quantum Limit, which is $\delta_{0,0}$.

Now, using that the spectrum of B_α has gaps that are uniformly bounded below, this property is preserved when we consider eigenfunctions of $-\Delta_j$: when α varies, the projection Φ_k^j of $(A_\alpha^*)^{n_j} f_{j,k}e^{i\alpha z}$ onto the eigenspace of $-\Delta_j$ corresponding to the eigenvalue $(2n_j+1)|\alpha|$ has a unique QL, which is $\mathscr{H}_0^s \otimes \delta_p$. The Dirac mass at p comes from (38) and from Lemma 28 applied, for any $1 \leq i, j \leq J$, to the operator $\frac{R_i}{R_j} - \frac{p_i}{p_j}$. Note that the point $q = 0$ plays no specific role, and therefore any measure $\mathscr{H}_q^s \otimes \delta_p$ can be obtained as a QL, when $d_q^s = 1$ and under (36). □

Lemma 22. *Let $(q, p) \in S\Sigma_{\mathcal{J}}$ and $s \in \mathbf{S}_{\mathcal{J}}$ be arbitrary. Then, the measure $\mathscr{H}_q^s \otimes \delta_p$ is a Quantum Limit. [See Remark 23 for the description of the associated sequence of normalized eigenfunctions.]*

Proof. We still assume that $\mathcal{J} = \{1, \dots, J\}$. Using Lemma 21, we can assume that $q \in \mathbf{H}^m$ and $s \in \mathbf{S}_j$ verify either $d_q^s \geq 2$, or $d_q^s = 1$ but (36) is not satisfied. In both cases, the following fact holds:

Fact 1. The measure $\mathscr{H}_q^s$ is in the weak-star closure of the set of measures $\mathscr{H}_{q'}^{s'}$ for which $d_{q'}^{s'} = 1$ and (36) is satisfied.

Let us denote by $\mathbb{T}^{\mathcal{J}} = (\mathbb{R}/2\pi\mathbb{Z})^{\mathcal{J}}$ the Riemannian torus of dimension $\#\mathcal{J}$ equipped with the flat metric. Due to Remark 7, proving Fact 1 is equivalent to proving the following fact, called Fact 2 in the sequel: if γ is a geodesic of $\mathbb{T}^{\mathcal{J}}$ and $\mathscr{H}_\gamma$ is the Hausdorff measure on γ, then $\mathscr{H}_\gamma$ is in the weak-star closure of the set of measures $\mathscr{H}_{\gamma'}$ with γ' a periodic geodesic of $\mathbb{T}^{\mathcal{J}}$ of slope $(s_1, \dots, s_J)$ verifying (36) for some J-tuple $(n_1, \dots, n_J)$. Let us prove Fact 2.

In case $d_q^s \geq 2$, possibly restricting to the flat torus given by the closure of γ, we can assume that γ is a dense geodesic in $\mathbb{T}^{\mathcal{J}}$. To prove Fact 2 in this elementary case, we take a sequence of geodesics $(\gamma_n')_{n \in \mathbb{N}^*}$ contained in $\mathbb{T}^{\mathcal{J}}$, with rational slopes given by J-tuples $(s_1^n, \dots, s_J^n)$ of the form (36), and which become dense in $\mathbb{T}^{\mathcal{J}}$ as $n \to +\infty$.

For the case $d_q^s = 1$ where (36) is not satisfied, similarly, we take a sequence of geodesics with rational slopes which converges to γ. This proves Fact 2 and hence Fact 1 follows.

Since the set of QLs is closed, Fact 1 implies Lemma 22. □

Remark 23. *Note that, following the proofs of Lemma 21 and Lemma 22, any measure $\mathscr{H}_q^s \otimes \delta_p$ is a Quantum Limit associated to a sequence of normalized eigenfunctions $(\varphi_k)_{k \in \mathbb{N}^*}$ such that, for any $k \in \mathbb{N}^*$, φ_k belongs to some eigenspace $\mathcal{H}^{\mathcal{J}}_{(n_{j,k}),(\alpha_{j,k})}$. In particular, φ_k is an eigenfunction of Ω_j for any $j \in \mathcal{J}$.*

Note also that to guarantee this last property, it is not sufficient to invoke, at the end of the proof of Lemma 22, the closedness of the set of QLs: it is necessary to follow the proof of this fact, which consists in a simple extraction argument.

Lemma 24. *Let $s \in \mathbf{S}_{\mathcal{J}}$ and $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$ having no mass outside $S\Sigma_{\mathcal{J}}$ and being invariant under $\vec{\rho}_s^{\mathcal{J}}$. Then $\nu_s^{\mathcal{J}}$ is a Quantum Limit. [See Remark 25 for the description of the associated sequence of normalized eigenfunctions.]*

Proof. Let us consider the set $\mathscr{P}_s^{\mathcal{J}} \subset \mathscr{P}(S^*\mathbf{H}^m)$ of probability measures

$$\nu_s^{\mathcal{J}} = \sum_{(q_i, p_i) \in \mathscr{E}} \beta_i \mathscr{H}_{q_i}^s \otimes \delta_{p_i} \tag{40}$$

where i ranges over some finite set $\mathcal{F}$, $\mathscr{E}$ is a set of pairs $(q_i, p_i) \in S\Sigma$, and $\beta_i \in \mathbb{R}$.

We consider $\nu_s^{\mathcal{J}} \in \mathscr{P}_s^{\mathcal{J}}$ defined by (40). Note that if $i \neq i'$, either $\mathscr{H}_{q_i}^s \otimes \delta_{p_i} = \mathscr{H}_{q_i'}^s \otimes \delta_{p_i'}$, or the supports of $\mathscr{H}_{q_i}^s \otimes \delta_{p_i}$ and $\mathscr{H}_{q_i'}^s \otimes \delta_{p_i'}$ are disjoint. Therefore, possibly gathering terms in the above sum, we assume that the supports of $\mathscr{H}_{q_i}^s \otimes \delta_{p_i}$ and $\mathscr{H}_{q_i'}^s \otimes \delta_{p_i'}$ are disjoint as soon as $i \neq i'$.

For $i \in \mathcal{F}$, using Lemma 21 and Lemma 22, we consider a sequence of eigenfunctions $(\varphi_k^i)_{k \in \mathbb{N}^*}$ with eigenvalues $(\lambda_k^i)_{k \in \mathbb{N}^*}$ and whose unique QL is $\mathscr{H}_{q_i}^s \otimes \delta_{p_i}$. According to the proof of these lemmas (see also Remark 23), we can also assume that $\varphi_k^i \in \mathcal{H}^{\mathcal{J}}_{(n_{j,k}),(\alpha_{j,k}^i)}$ for some J-tuples such that

$$\lambda_k^i := \sum_{j \in \mathcal{J}} (2n_{j,k} + 1)|\alpha_{j,k}^i|$$

does not depend on $i \in \mathcal{F}$. In other words,

- for any $1 \leq j \leq J$, φ_k^i is also an eigenvalue of Ω_j with eigenvalue $n_{j,k}$ which does not depend on $i \in \mathcal{F}$;
- for any $i, i' \in \mathcal{F}$, $\lambda_k^i = \lambda_k^{i'}$ and we denote this common value by λ_k. This means that for any $i \in \mathcal{F}$, φ_k^i belongs to the eigenspace of $-\Delta$ corrresponding to the eigenvalue λ_k.

Since $\mathscr{H}^s_{q_i} \otimes \delta_{p_i}$ and $\mathscr{H}^s_{q_i'} \otimes \delta_{p_i'}$ have disjoint supports, the joint microlocal defect measure of $(\varphi_k^i)_{k\in\mathbb{N}^*}$ and $(\varphi_k^{i'})_{k\in\mathbb{N}^*}$ vanishes for $i \neq i'$ by Lemma 14. It follows that

$$\varphi_k := \sum_{i\in\mathcal{F}} \beta_i \varphi_k^i$$

is an eigenfunction of $-\Delta$ with eigenvalue λ_k, and that in the limit $k \to +\infty$, it admits $\nu_s^{\mathcal{J}}$ as unique Quantum Limit.

Finally, we note that any $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$ having all its mass contained in $S\Sigma_{\mathcal{J}}$ and being invariant under $\vec{\rho}_s^{\mathcal{J}}$ is in the closure of $\mathscr{P}_s^{\mathcal{J}}$. Since the set of QLs is closed, Lemma 24 is proved. □

Remark 25. *The above proof shows that $\nu_\infty = \nu_s^{\mathcal{J}}$ is a QL for a sequence $(\varphi_k)_{k\in\mathbb{N}^*}$ such that φ_k belongs to*

$$\bigoplus_{(\alpha_j)\in(\mathbb{Z}^*)^{\mathcal{J}}} \mathcal{H}^{\mathcal{J}}_{(n_{j',k'}),(\alpha_j)}$$

for some J-tuple $(n_{j',k'}) \in \mathbb{N}^{\mathcal{J}}$ which depends only on $k \in \mathbb{N}^*$.

Lemma 26. *Let $\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}$, and*

$$\nu^{\mathcal{J}} = \int_{\mathbf{S}_{\mathcal{J}}} \nu_s^{\mathcal{J}} dQ^{\mathcal{J}}(s)$$

for some $Q^{\mathcal{J}} \in \mathscr{P}(\mathbf{S}_{\mathcal{J}})$ and $\nu_s^{\mathcal{J}} \in \mathscr{P}(S^*\mathbf{H}^m)$ having no mass outside $S\Sigma_{\mathcal{J}}$ and such that, for $Q^{\mathcal{J}}$-almost any $s \in \mathbf{S}_{\mathcal{J}}$, $\vec{\rho}_s^{\mathcal{J}} \nu_s^{\mathcal{J}} = 0$. Then $\nu^{\mathcal{J}}$ is a Quantum Limit. [See Remark 27 for the description of the associated sequence of normalized eigenfunctions.]

Proof. As in the previous proofs, we assume without loss of generality that $\mathcal{J} = \{1, \ldots, J\}$ for some $1 \leq J \leq m$. Let $(s^\ell)_{\ell\in\mathcal{L}}$ be a finite family of distinct elements of $\mathbf{S}_{\mathcal{J}}$ indexed by $\mathcal{L}$, and let $\gamma_\ell \in \mathbb{R}$ for $\ell \in \mathcal{L}$. For any $\ell \in \mathcal{L}$, let also ν_{s^ℓ}, with mass only in $S\Sigma_{\mathcal{J}}$, be invariant under the flow of $\vec{\rho}^{\mathcal{J}}_{s^\ell}$. Let us prove that

$$\nu^{\mathcal{J}} = \sum_{\ell\in\mathcal{L}} \gamma^\ell \nu_{s^\ell} \tag{41}$$

is a Quantum Limit. This corresponds to the case where the measure $Q^{\mathcal{J}}$ on $\mathbf{S}_{\mathcal{J}}$ is given by

$$Q^{\mathcal{J}} = \sum_{\ell\in\mathcal{L}} \gamma^\ell \delta_{s^\ell}.$$

For any $\ell \in \mathcal{L}$, we take $(\varphi_k^\ell)_{k\in\mathbb{N}^*}$ to be a sequence of eigenfunctions of $-\Delta$ whose unique QL is ν_{s^ℓ}. As emphasized in the proof of Lemma 24, it is possible to assume that φ_k^ℓ is an eigenfunction of Ω_j for any $1 \leq j \leq J$, with eigenvalue $2n_{j,k}^\ell + 1$ such that

$$\frac{2n_{j,k}^\ell + 1}{\sum_{i=1}^{J}(2n_{i,k}^\ell) + 1} \underset{k\to+\infty}{\longrightarrow} s_j^\ell \tag{42}$$

where $s^\ell = (s_1^\ell, \ldots, s_J^\ell)$.

Let us prove that the joint microlocal defect measure $\nu_{\ell,\ell'}$ of $(\varphi_k^\ell)_{k\in\mathbb{N}^*}$ and $(\varphi_k^{\ell'})_{k\in\mathbb{N}^*}$ vanishes for $\ell \neq \ell'$: we note that for $\mathrm{Op}(a)$ commuting with $\Omega_1, \ldots, \Omega_m$, with $a \in \mathscr{S}^0(\mathbf{H}^m)$,

$$\begin{aligned}(2n_{j,k}^\ell + 1)(\mathrm{Op}(a)\varphi_k^\ell, \varphi_k^{\ell'}) &= (\mathrm{Op}(a)\Omega_j\varphi_k^\ell, \varphi_k^{\ell'}) \\ &= (\mathrm{Op}(a)\varphi_k^\ell, \Omega_j\varphi_k^{\ell'}) \\ &= (2n_{j,k}^{\ell'} + 1)(\mathrm{Op}(a)\varphi_k^\ell, \varphi_k^{\ell'})\end{aligned}$$

From (42) and the fact that $s^\ell \neq s^{\ell'}$, we deduce that, for any sufficiently large $k \in \mathbb{N}^*$, there exists $1 \leq j \leq J$ such that $n^\ell_{j,k} \neq n^{\ell'}_{j,k}$. Hence, the above computation shows that $(\mathrm{Op}(a)\varphi^\ell_k, \varphi^{\ell'}_k) = 0$ for sufficiently large $k \in \mathbb{N}^*$. Therefore,

$$\int_{S^*\mathbf{H}^m} a d\nu_{\ell,\ell'} = 0.$$

Since ν_{s^ℓ} and $\nu_{s^{\ell'}}$ give no mass to the complementary set of $S\Sigma_{\mathcal{J}}$ in $S^*\mathbf{H}^m$, we know that it is also the case for $\nu_{\ell,\ell'}$ by Lemma 14. Therefore, if $b \in \mathscr{S}^0(\mathbf{H}^m)$ is arbitrary, averaging $\mathrm{Op}(b)$ with respect to the operators $\Omega_1, \ldots, \Omega_J$ as in Lemma 18, we obtain an operator $A \in \Psi^0(\mathbf{H}^m)$ such that $\sigma_P(A)$ coincides with b on $\Sigma_{\mathcal{J}}$, and A commutes with $\Omega_1, \ldots, \Omega_J$. Therefore,

$$\int_{S^*\mathbf{H}^m} b d\nu_{\ell,\ell'} = \int_{S\Sigma_{\mathcal{J}}} b d\nu_{\ell,\ell'} = \int_{S\Sigma_{\mathcal{J}}} \sigma_P(A) d\nu_{\ell,\ell'} = 0,$$

and since this is true for any $b \in \mathscr{S}^0(\mathbf{H}^m)$, we conclude that $\nu_{\ell,\ell'} = 0$.

This implies that the sequence given by

$$\varphi^{\mathcal{J}}_k = \sum_{\ell \in \mathcal{L}} \gamma^\ell \varphi^\ell_k$$

admits $\nu^{\mathcal{J}}$ as unique QL, where $\nu^{\mathcal{J}}$ is defined by (41). Note that to ensure that $\varphi^{\mathcal{J}}_k$ is still an eigenfunction of $-\Delta$, it is necessary, as in the proof of Lemma 24, to adjust the sequences $(n^\ell_{j,k})$ and $(\alpha^\ell_{j,k})$ in order to guarantee that all φ^ℓ_k (for $\ell \in \mathcal{L}$) are eigenfunctions of $-\Delta$ with same eigenvalue.

We notice that the closure of the set of Radon measures on $S\Sigma_{\mathcal{J}}$ which may be written as a finite linear combination (41) is exactly the subset of $\mathscr{P}_{S\Sigma}$ for which $Q^{\mathcal{J}'} = 0$ for any $\mathcal{J}' \neq \mathcal{J}$. Using that the set of QLs is closed, Lemma 26 is proved. □

Remark 27. *The above proof shows that $\nu_\infty = \nu^{\mathcal{J}}$ is a QL for a sequence of normalized eigenfunctions $(\varphi_k)_{k \in \mathbb{N}^*}$ such that φ_k belongs to*

$$\bigoplus_{(n_j) \in \mathbb{N}^{\mathcal{J}}} \bigoplus_{(\alpha_j) \in (\mathbb{Z}^*)^{\mathcal{J}}} \mathcal{H}^{\mathcal{J}}_{(n_j),(\alpha_j)}.$$

Let us now finish the proof of Theorem 3. Let $\nu_\infty \in \mathscr{P}_{S\Sigma}$,

$$\nu_\infty = \sum_{\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}} \nu^{\mathcal{J}}.$$

Note that the measures $\nu^{\mathcal{J}}$ are non-negative, but are not necessarily probability measures.

Let $(\varphi^{\mathcal{J}}_k)_{k \in \mathbb{N}^*}$ be a sequence of eigenfunctions of $-\Delta$ whose unique microlocal defect measure is $\nu^{\mathcal{J}}$. The proof of Lemma 26 guarantees that, for any $k \in \mathbb{N}^*$, one may choose all $\varphi^{\mathcal{J}}_k$, for $\mathcal{J}$ running over $\mathcal{P} \setminus \{\emptyset\}$, to have the same eigenvalue with respect to $-\Delta$. Therefore,

$$\varphi_k = \sum_{\mathcal{J} \in \mathcal{P} \setminus \{\emptyset\}} \varphi^{\mathcal{J}}_k$$

is also an eigenfunction of $-\Delta$. Moreover, for any distinct $\mathcal{J}, \mathcal{J}' \in \mathcal{P} \setminus \{\emptyset\}$, the joint microlocal defect measure of $(\varphi^{\mathcal{J}}_k)_{k \in \mathbb{N}^*}$ and $(\varphi^{\mathcal{J}'}_k)_{k \in \mathbb{N}^*}$ vanishes (see Lemma 14). Computing $(\mathrm{Op}(a)\varphi_k, \varphi_k)$ for any $a \in \mathscr{S}^0(\mathbf{H}^m)$ in the limit $k \to +\infty$, we obtain that the unique Quantum Limit of $(\varphi_k)_{k \in \mathbb{N}^*}$ is ν_∞. Note that, as already explained in Remarks 23, 25 and 27, the sequence $(\varphi_k)_{k \in \mathbb{N}^*}$ is fully explicit in our construction.

Finally, we note that the invariance properties of ν_∞ can be established separately on each $S\Sigma_{\mathcal{J}}$ since $([A, R_s]\varphi^{\mathcal{J}}_k, \varphi^{\mathcal{J}'}_k) \to 0$ as $k \to +\infty$ for $\mathcal{J} \neq \mathcal{J}'$ (the bracket $[A, R_s]$ is the natural operator to consider for establishing invariance properties, see Section 3.3). This concludes the proof of Theorem 3.

A Classical pseudodifferential calculus

We briefly gather some basic facts of pseudodifferential calculus used along this paper (see also [Hör85, Chapter XVIII]).

Following our notations of Section 1, we denote by M a smooth compact manifold of dimension n. We denote by $\mathscr{S}^k(M)$ the space of smooth homogeneous functions of order k on the cone $T^*M \setminus \{0\}$. They are the *classical symbols* of order k.

The algebra $\Psi(M)$ of classical pseudodifferential operators on M is graded according to the chain of inclusions $\Psi^{-\infty}(M) \subset \ldots \subset \Psi^k(M) \subset \Psi^{k+1}(M) \subset \ldots$ where $k \in \mathbb{Z} \cup \{-\infty\}$ is called the order.

To a pseudodifferential operator $A \in \Psi^m(M)$, we can associate its *principal symbol* $\sigma_P(A)$, and the map $\sigma_P : \Psi^k(M)/\Psi^{k-1}(M) \to \mathscr{S}^k(M)$ is bijective. A *quantization* is a continuous linear mapping

$$\mathrm{Op} : \mathscr{S}^0(M) \to \Psi^0(M)$$

with $\sigma_P(\mathrm{Op}(a)) = a$. An example is obtained using partitions of unity and the Weyl quantization which is given in local coordinates by

$$\mathrm{Op}^W(a)f(q) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\langle q-q',p\rangle} a\left(\frac{q+q'}{2}, p\right) f(q')dq'dp.$$

Although we omitted the upper W index in the paper, this is the quantization we used by default in this paper.

We have the following properties:

- If $A \in \Psi^k(M)$ and $B \in \Psi^\ell(M)$, then $AB \in \Psi^{k+\ell}(M)$ and $\sigma_P(AB) = \sigma_P(A)\sigma_P(B)$.
- If $A \in \Psi^k(M)$ and $B \in \Psi^\ell(M)$, then $[A, B] \in \Psi^{k+\ell-1}(M)$ and
$$\sigma_P([A, B]) = \frac{1}{i}\{\sigma_P(A), \sigma_P(B)\},$$
where the Poisson bracket is taken with respect to the canonical symplectic structure of T^*M.

Let us prove Lemma 14 of Section 2.

Proof of Lemma 14. If $a \in \mathscr{S}^0(M)$ is such that $a \geq 0$ and a is supported in a set where $\mu_{11} = 0$, then, setting $a_\varepsilon = a + \varepsilon$ for any $\varepsilon > 0$, we get

$$(\mathrm{Op}(a_\varepsilon)u_k, v_k) = (\mathrm{Op}(a_\varepsilon^{1/2})u_k, \mathrm{Op}(a_\varepsilon^{1/2})v_k) + o(1) \leq \|\mathrm{Op}(a_\varepsilon^{1/2})u_k\|_{L^2}\|\mathrm{Op}(a_\varepsilon^{1/2})v_k\|_{L^2} + o(1)$$

where $a_\varepsilon^{1/2} \in \mathscr{S}^0(M)$. We know that

$$\|\mathrm{Op}(a_\varepsilon^{1/2})u_k\|_{L^2}^2 = (\mathrm{Op}(a_\varepsilon)u_k, u_k) + o(1) = (\mathrm{Op}(a)u_k, u_k) + \varepsilon\|u_k\|_2^2 + o(1) = \varepsilon\|u_k\|^2 + o(1)$$

and that $\|\mathrm{Op}(a_\varepsilon^{1/2})v_k\|_{L^2}^2 \leq (C + \varepsilon)\|v_k\|^2$ where C does not depend on ε. Therefore $(\mathrm{Op}(a_\varepsilon)u_k, v_k) \lesssim \varepsilon$. Hence $(\mathrm{Op}(a)u_k, v_k) \to 0$. The same result holds for $a \leq 0$ supported in a set where $\mu_{11} = 0$. Therefore, decomposing any symbol as $a = a^+ + a^- + r$, where $a^+, a^-, r \in \mathscr{S}^0(M)$, $a^+ \geq 0$, $a^- \leq 0$, and $|r| \leq \delta$ for some small $\delta > 0$, we get that μ_{12} is absolutely continuous with respect to μ_{11}. The rest of the lemma follows by symmetry. $\square$

Lemma 28. *Let us assume that $\ell \in \mathbb{N}$ and $P \in \Psi^\ell(M)$ is elliptic in any cone contained in the complementary of a closed conic set $F \subset T^*M$. Assume that $(u_k)_{k\in\mathbb{N}^*}$ is a bounded sequence in $L^2(M)$ weakly converging to 0 and such that $Pu_k \to 0$ strongly in $L^2(M)$. Then any microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$ is supported in F.*

Proof. Let μ be a microlocal defect measure of $(u_k)_{k\in\mathbb{N}^*}$, i.e.,

$$(\mathrm{Op}(a)u_{\sigma(k)}, u_{\sigma(k)}) \underset{k\to+\infty}{\longrightarrow} \int_{S^*M} a d\mu$$

for any $a \in \mathscr{S}^0(M)$, where σ is an extraction. Let $a \in \mathscr{S}^0(M)$ be supported outside F. Let $Q \in \Psi^{-\ell}(M)$ be such that $PQ - I \in \Psi^{-1}(M)$ on the support of a. Then $Q\mathrm{Op}(a)P \in \Psi^0(M)$ has principal symbol a, and therefore

$$(Q\mathrm{Op}(a)Pu_{\sigma(k)}, u_{\sigma(k)}) \underset{k\to+\infty}{\longrightarrow} \int_{S^*M} ad\mu.$$

Using that $Pu_{\sigma(k)} \to 0$, we get $(Q\mathrm{Op}(a)Pu_{\sigma(k)}, u_{\sigma(k)}) \to 0$ as $k \to +\infty$, and therefore $\int_{S^*M} ad\mu = 0$. Hence, μ is supported in F. □

B Supplementary material on Assumption (A)

B.1 H-type sub-Laplacians

Let us explain briefly what we mean by H-type sub-Laplacians, which are examples satisfying Assumption (A). An H-type group G is a connected and simply connected Lie group whose Lie algebra is H-type: its Lie algebra $\mathfrak{g}$ is equipped with a vector space decomposition

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z} ,$$

such that $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z} \neq \{0\}$ and $\mathfrak{z}$ is the center of $\mathfrak{g}$. We say that $\mathfrak{g}$ is H-type if it can be endowed with a scalar product $\langle \cdot, \cdot \rangle$ such that, for all $\lambda \in \mathfrak{z}^*$, the skew-symmetric map

$$J_\lambda : \mathfrak{v} \to \mathfrak{v}$$

defined by

$$\langle J_\lambda(U), V \rangle = \lambda([U, V]) \quad \forall U, V \in \mathfrak{v}$$

satisfies $J_\lambda^2 = -|\lambda|^2\mathrm{Id}$. Consider Γ a discrete cocompact subgroup of G, and $M = \Gamma \backslash G$. Let us choose an orthonormal basis V_j of $\mathfrak{v}$ and identify $\mathfrak{g}$ with the Lie algebra of left-invariant vector fields on G. Then, we can consider the sub-Laplacian

$$\Delta_M = \sum_{j=1}^{2d} V_j^2,$$

on M, where $\dim \mathfrak{v} = 2d$. Note that this makes sense since the V_j are left-invariant, and thus pass to the quotient. This is what we call an H-type sub-Laplacian. An example is given by sub-Laplacians on the Heisenberg group in any odd dimension, which we now define.

For $d \geq 1$, we consider the group law on $\mathbb{R}^{2d+1}$ given by

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' - x \cdot y')$$

where $x, x', y, y' \in \mathbb{R}^d$ and $z, z' \in \mathbb{R}$. The Heisenberg group $\widetilde{\mathbf{H}}_d$ is the group $\widetilde{\mathbf{H}}_d = (\mathbb{R}^{2d+1}, \star)$. We consider the subgroup $\Gamma_d = (\sqrt{2\pi}\mathbb{Z})^{2d} \times 2\pi\mathbb{Z}$ of $\widetilde{\mathbf{H}}_d$, and the left quotient $\mathbf{H}_d = \Gamma_d \backslash \widetilde{\mathbf{H}}_d$. We also define the $2d$ left invariant vector fields on $\mathbf{H}_d$ given by

$$X_j = \partial_{x_j}, \qquad Y_j = \partial_{y_j} - x_j \partial_z$$

for $1 \leq j \leq d$. We fix $\beta_1, \ldots, \beta_d > 0$ satisfying $\prod_{j=1}^d \beta_j = 1$, we set $\beta = (\beta_1, \ldots, \beta_d)$ and we consider the sub-Laplacian

$$\Delta_\beta = \sum_{j=1}^{d} \beta_j (X_j^2 + Y_j^2) \tag{43}$$

which is an operator acting on functions on $\mathbf{H}_d$. The positive real numbers β_j are sometimes called frequencies, see [Agr96].

B.2 The Martinet sub-Laplacian

In this Section, we provide an example of a sub-Laplacian on a compact manifold which satisfies Assumption (A) but which is not step 2, meaning that brackets of length ≥ 3 of the X_i are required to generate the whole tangent bundle, see (1).

For that, we consider $M = (\mathbb{R}/2\pi\mathbb{Z})^3$ with coordinates x, y, z, endowed with the Lebesgue measure $d\mu = dxdydz$. Let A be a smooth 1-form $A = A_x dx + A_y dy$, where A_x and A_y depend only on x and y. The 2-form $B = dA = (\partial_x A_y - \partial_y A_x) dx \wedge dy$ is the "magnetic field" and $b = \partial_x A_y - \partial_y A_x$ is its "strength". We consider the sub-Riemannian structure associated to the vector fields $X_1 = \partial_x + A_x \partial_z$ and $X_2 = \partial_y + A_y \partial_z$. Then, $[X_1, X_2] = b\partial_z$. Now, we choose A so that b vanishes along a closed curve in $(\mathbb{R}/2\pi\mathbb{Z})^2_{x,y}$, and $(\partial_x b, \partial_y b) \neq 0$ along this curve. This construction is classical, see [Mon95]. When adding the z-variable, this yields a surface $\mathscr{S} \subset M$, called Martinet surface, on which $[X_1, X_2] = 0$ but some bracket of length 3 of X_1, X_2 generates the missing direction of the tangent bundle thanks to $(\partial_x b, \partial_y b) \neq 0$. In other words, the sub-Riemannian structure has step 3 on $\mathscr{S}$. Nevertheless, Assumption (A) is satisfied with $Z_1 = \partial_z$.

C Quantum Limits of flat contact manifolds

The study of Quantum Limits of higher dimensional contact manifolds is also an interesting problem. In this section, we prove that for the sub-Laplacian (43) defined on the quotient of the Heisenberg group $\mathbf{H}_d$ of dimension $2d+1$ by one of its discrete cocompact subgroups, the invariance properties of Quantum Limits are much simpler than those described in Theorem 2, even though "frequencies" show up: the part of the QL which lies in $S\Sigma$ is invariant under the lift of the Reeb flow, as in the 3D case. We borrow the notations from Appendix B.1.

We set $\rho = h_Z|_\Sigma$, which is the Hamiltonian lift of the Reeb vector field $Z = \partial_z$ to Σ (see [CdVHT18, Section 2.3] for properties of the Reeb vector field).

Proposition 29. *Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be a sequence of $L^2(\mathbf{H}_d)$ consisting of normalized eigenfunctions of $-\Delta_\beta$. Then, any Quantum Limit ν_∞ associated to $(\varphi_k)_{k\in\mathbb{N}^*}$ and supported in $S\Sigma$ is invariant under $e^{t\vec{\rho}}$, the lift of the Reeb flow.*

Remark 30. *This result follows from [FKF20, Theorem 2.10(ii)(2)], but we provide here a simple self-contained proof which illustrates the averaging techniques used in Section 3.3.*

Remark 31. *We do not expect such a result to be true when the frequencies β_j are not constant on the manifold.*

Proof of Proposition 29. Denoting by (q, p) the canonical coordinates in $T^*\mathbf{H}_d$, i.e., $q = (x_1, \ldots, x_d, y_1, \ldots, y_d, z)$ and $p = (p_{x_1}, \ldots, p_{x_d}, p_{y_1}, \ldots, p_{y_d}, p_z)$, we know that

$$\Sigma = \{(q, p) \in T^*\mathbf{H}_d,\ p_{x_j} = p_{y_j} - x_j p_{z_j} = 0\}$$

is isomorphic to $\mathbf{H}_d \times \mathbb{R}$.

Up to extraction of a subsequence, we may assume that $(\varphi_k)_{k\in\mathbb{N}^*}$ has a unique QL ν_∞, which is supported in $S\Sigma$. We set $R = \sqrt{\partial_z^* \partial_z}$ and, on its eigenspaces corresponding to non-zero eigenvalues, we define $\Omega_j = -R^{-1}(X_j^2 + Y_j^2) = -(X_j^2 + Y_j^2)R^{-1}$ for $1 \leq j \leq d$. On these eigenspaces, the sub-Laplacian acts as

$$-\Delta_\beta = R\Omega = \Omega R \qquad \text{with} \quad \Omega = \sum_{j=1}^{d} \beta_j \Omega_j$$

and $[R, \Omega] = 0$.

Let V be a (small) conic microlocal neighborhood of Σ, and let us consider R, Ω as acting on functions microlocally supported in V (meaning that their wave-front set is contained in

V). If $B \in \Psi^0(\mathbf{H}_d)$ is microlocally supported in V and commutes with Ω, then

$$\begin{aligned}
([B,R]\varphi_k,\varphi_k) &= \frac{1}{\lambda_k}(BR\varphi_k, -\Delta_\beta\varphi_k) - \frac{1}{\lambda_k}(RB(-\Delta_\beta)\varphi_k,\varphi_k)\\
&= \frac{1}{\lambda_k}(BR\varphi_k, R\Omega\varphi_k) - \frac{1}{\lambda_k}(RBR\Omega\varphi_k,\varphi_k)\\
&= \frac{1}{\lambda_k}([\Omega, RBR]\varphi_k,\varphi_k)\\
&= 0.
\end{aligned}$$

Let $U(t) = U(t_1,\ldots,t_d) = e^{i(t_1\Omega_1+\ldots+t_d\Omega_d)}$ for $t=(t_1,\ldots,t_d)\in(\mathbb{R}/2\pi\mathbb{Z})^d$. For $A \in \Psi^0(\mathbf{H}_d)$ microlocally supported in V, we consider

$$\widetilde{A} = \int_{(\mathbb{R}/2\pi\mathbb{Z})^d} U(-t)AU(t)dt$$

As in the proof of Lemma 18, we know that $[\widetilde{A},\Omega]=0$ and that $\sigma_P(A)$ and $\sigma_P(\widetilde{A})$ coincide on Σ. Therefore, using the previous computation with $B=\widetilde{A}$, we obtain

$$\int_\Sigma \{\sigma_P(A),\rho\}_{\omega_{|\Sigma}} d\nu_\infty = \int_\Sigma \{\sigma_P(\widetilde{A}),\rho\}_{\omega_{|\Sigma}} d\nu_\infty = \lim_{k\to+\infty}([\widetilde{A},R]\varphi_k,\varphi_k) = 0.$$

Since it is true for any A microlocally supported in V, this implies that ν_∞ is invariant under the flow $e^{t\vec{\rho}}$. □

References

[ABB19] Andrei Agrachev, Davide Barilari, and Ugo Boscain. A comprehensive introduction to sub-Riemannian geometry. *Cambridge University Press, volume 181,* 2019.

[Agr96] Andrei Agrachev. Exponential mappings for contact sub-Riemannian structures. *Journal of dynamical and control systems*, 2(3):321–358, 1996.

[ALM16] Nalini Anantharaman, Matthieu Léautaud, and Fabricio Macià. Wigner measures and observability for the Schrödinger equation on the disk. *Inventiones mathematicae*, 206(2):485–599, 2016.

[BFKG] Hajer Bahouri, Clotilde Fermanian Kammerer, and Isabelle Gallagher. Dispersive estimates for the Schrödinger operator on step-2 stratified Lie groups. Analysis & PDE, 9(3):545-574, 2016.

[CdV79] Yves Colin de Verdière. Spectre conjoint d'opérateurs pseudodifférentiels qui commutent, I. Le cas non intégrable. *Duke mathematical journal*, 46(1):169–182, 1979.

[CdV84] Yves Colin de Verdière. Calcul du spectre de certaines nilvariétés compactes de dimension 3. *Séminaire de théorie spectrale et géométrie*, 2:1–6, 1983-1984.

[CdV85] Yves Colin de Verdière. Ergodicité et fonctions propres du laplacien. *Communications in Mathematical Physics*, 102(3):497–502, 1985.

[CdVHT18] Yves Colin de Verdière, Luc Hillairet, and Emmanuel Trélat. Spectral asymptotics for sub-Riemannian Laplacians, I: Quantum ergodicity and quantum limits in the 3-dimensional contact case. *Duke Mathematical Journal*, 167(1):109–174, 2018.

[DS84] Christopher Deninger and Wilhelm Singhof. The e-invariant and the spectrum of the Laplacian for compact nilmanifolds covered by Heisenberg groups. *Inventiones Mathematicae*, 78(1):101–112, 1984.

[FK00] Clotilde Fermanian Kammerer. Mesures semi-classiques 2-microlocales. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 331(7):515–518, 2000.

[FKF20] Clotilde Fermanian Kammerer and Véronique Fischer. Quantum evolution and sub-Laplacian operators on groups of Heisenberg type. *To appear in Journal of Spectral Theory*, 2020.

[FKL20] Clotilde Fermanian Kammerer and Cyril Letrouit. Observability and controllability for the Schrödinger equation on quotients of groups of Heisenberg type. *ArXiv preprint arXiv:2009.13877*, 2020.

[Gér91a] Patrick Gérard. Mesures semi-classiques et ondes de Bloch. *Séminaire Équations aux dérivées partielles (Polytechnique)*, pages 1–19, 1991.

[Gér91b] Patrick Gérard. Microlocal defect measures. *Communications in Partial differential equations*, 16(11):1761–1794, 1991.

[Hör67] Lars Hörmander. Hypoelliptic second order differential equations. *Acta Mathematica*, 119:147–171, 1967.

[Hör85] Lars Hörmander. The analysis of linear partial differential operators. III, volume 274 of Grundlehren der Mathematischen Wissenschaften, 1985.

[HPT20] Emmanuel Humbert, Yannick Privat, and Emmanuel Trélat. Observability for generalized Schrödinger equations and Quantum Limits on product manifolds. *ArXiv preprint arXiv:2003.03094*, 2020.

[HV00] Andrew Hassell and András Vasy. Symbolic functional calculus and N-body resolvent estimates. *Journal of Functional Analysis*, 173(2):257–283, 2000.

[Jak97] Dmitry Jakobson. Quantum limits on flat tori. *Annals of mathematics*, pages 235–266, 1997.

[JZ96] Dmitry Jakobson and Steve Zelditch. Classical limits of eigenfunctions for some completely integrable systems. *Emerging applications of number theory (Minneapolis, MN, 1996), IMA Vol. Math. Appl*, 109:329–354, 1999.

[Kap80] A. Kaplan. Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Transactions of the American Mathematical Society*, 258(1): 147–153, 1980.

[Lin06] Elon Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Annals of Mathematics*, pages 165–219, 2006.

[Mon95] Richard Montgomery. Hearing the zero locus of a magnetic field. *Communications in Mathematical Physics, 168:651–675, 1995.*

[Mon02] Richard Montgomery. A tour of subriemannian geometries, their geodesics and applications. *Number 91. American Mathematical Soc.*, 2002.

[MRS95] Detlef Müller, Fulvio Ricci and Elias Stein. Marcinkiewicz multipliers and multiparameter structure on Heisenberg(-type) groups, I. *Inventiones Mathematicae*, 119(2):199–233, 1995.

[RS72] Michael Reed and Barry Simon. Methods of modern mathematical physics, I. Functional analysis. *Academic Press, New York-London*, 1972.

[Roh62] Vladimir Abramovitch Rohlin. On the fundamental concepts of measure theory. *Amer. Math. Soc. Transl.*, 1(10), 1962.

[Shn74] Alexander I. Shnirel'man. Ergodic properties of eigenfunctions. *Uspekhi Matematicheskikh Nauk*, 29(6):181–182, 1974.

[Str91] Robert Strichartz. L^p harmonic analysis and Radon transforms on the Heisenberg group. *Journal of functional Analysis*, 96(2):350–406, 1991.

[Stu19] Elie Studnia. Quantum Limits for Harmonic Oscillator. *ArXiv preprint arXiv:1905.07763*, 2019.

[Tay86] Michael Taylor. Noncommutative harmonic analysis. *volume 22 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI*, 1986.

[Tha09] Sundaram Thangavelu. Harmonic analysis on Heisenberg nilmanifolds. *Rev. Un. Mat. Argentina*, 50(2):75–93, 2009.

[Wei77] Alan Weinstein. Asymptotics of eigenvalue clusters for the Laplacian plus a potential. *Duke Mathematical Journal*, 44(4):883–892, 1977.

[Zel87] Steven Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke mathematical journal*, 55(4):919–941, 1987.

[Zwo12] Maciej Zworski. Semiclassical analysis. *Number 138. American Mathematical Soc.*, 2012.